Residual a posteriori error estimations in one dimension

Let $\Omega =]a, b[$ be a non-empty bounded open subset of $\mathbb{R}$. For a function $f \in L^2(\Omega)$, we consider the Laplace equation

$$\begin{aligned} -u'' &= f \ \text{ in } \Omega \\ u &= 0 \ \text{ on } \partial\Omega \end{aligned}$$

We discretize this problem in the usual way: we introduce real numbers x_i such that

$$a = x_0 < x_1 < \cdots < x_i < \cdots < x_N = b,$$

then denote by I_i the interval $]x_{i-1}, x_i[$ for $1 \leq i \leq N$, and let h_i be its length. As usual, the parameter h is the maximum of h_i, $1 \leq i \leq N$. For a fixed integer $k \geq 1$, we introduce the discrete space

$$V_h = \left\{ v_h \in C^0(\overline{\Omega});\ v_h|_{I_i} \in P_k(I_i), 1 \leq i \leq N \right\} \cap H_0^1(\Omega),$$

where $P_k(I_i)$ is the space of polynomials of degree $\leq k$ on I_i.
The discrete problem is written as:

$$\text{Find } u_h \in V_h \text{ such that} \int_a^b u_h'(x) v_h'(x)\, dx = \int_a^b f(x) v_h(x)\, dx, \quad \forall v_h \in V_h. \tag{1}$$

Next, we define the family of indicators $(\eta_i)_{1 \leq i \leq N}$ by

$$\eta_i = h_i \| f_h + u_h'' \|_{L^2(I_i)},$$

where f_h is an approximation of f whose restriction to each I_i belongs to $P_{\max(k-2,0)}(I_i)$.
Let κ_1 and κ_2 be the smallest constants such that

$$|u - u_h|_{H^1(\Omega)} \leq \kappa_1 \left(\sum_{j=1}^N \eta_i^2 \right)^{1/2} + c \| f - f_h \|_{L^2(\Omega)},$$

$$\eta_i \leq \kappa_2 |u - u_h|_{H^1(I_i)} + c' \| f - f_h \|_{L^2(I_i)}.$$

We aim to establish an explicit upper bound for κ_1.

Question 1

We introduce an operator τ_h from $H_0^1(\Omega)$ into V_h such that:

$$\forall v \in H_0^1(\Omega), (\tau_h v)(x_i) = v(x_i), \quad 0 \leq i \leq N.$$

Show that:

$$|u - u_h|_{H^1(\Omega)} \leq \sup_{v \in H_0^1(\Omega)} \sum_{i=1}^N \left(\| f_h + u_h'' \|_{L^2(I_i)} + \| f - f_h \|_{L^2(I_i)} \right) \frac{\| v - \tau_h v \|_{L^2(I_i)}}{|v|_{H^1(\Omega)}}.$$

Question 2

In the case where $k = 1$, verify that for any function $v \in H_0^1(\Omega)$, $\tau_h v$ on each I_i is given by

$$(\tau_h v)(x) = v(x_{i-1})\frac{x_i - x}{h_i} + v(x_i)\frac{x - x_{i-1}}{h_i}.$$

Hint: Use the usual Taylor formula to show that

$$v(x) = (\tau_h v)(x) + \frac{x_i - x}{h_i}\int_{x_{i-1}}^{x} v'(t)\,dt - \frac{x - x_{i-1}}{h_i}\int_{x}^{x_i} v'(t)\,dt.$$

Question 3

Deduce an upper bound for $\|v - \tau_h v\|_{L^2(I_i)}$ in terms of $|v|_{H^1(I_i)}$ when $k = 1$. Hint: Use the formula established in Question 2 and the Cauchy-Schwarz inequality to show that

$$\|v - \tau_h v\|_{L^2(I_i)} \leq h_i \frac{\sqrt{3}}{3} |v|_{H^1(I_i)}.$$

Question 4

In the case where $k = 1$, provide an upper bound for κ_1.