Exercises

Ex.1. We consider the following boundary value problem:

$$-(\alpha u')' + \beta u' + \gamma u = f \quad \text{in } (a,b) \tag{1}$$
$$u(a) = u(b) = 0 \tag{2}$$

where $\alpha > 0$, $\beta \in \mathbb{R}$, $\gamma \geqslant 0$, and $f : (a,b) \to \mathbb{R}$ is a sufficiently regular function.

(a) Write the problem in weak form (variational form), that is, in the form:

$$\begin{cases} \text{Find } u \in V \text{ such that :} \\ a(u,v) = F(v) \quad \forall v \in V \end{cases} \tag{3}$$

Specify the definition of the bilinear form $a(\cdot,\cdot)$ and the linear functional $F(\cdot)$.

Ex.2. We consider the following boundary value problem with non-homogeneous boundary conditions:

$$-(\alpha u')' + \beta u' + \gamma u = f \quad \text{in } (0,1) \tag{4}$$
$$u(0) = 1, \quad u(1) = 2 \tag{5}$$

We observe that the technique used in Exercise 1 to write the weak form of the given problem no longer works if applied as is. The goal of this exercise is to reduce the problem to one with homogeneous boundary conditions. To do so, define a function $R : (0,1) \to \mathbb{R}$, called the lifting function, which satisfies the given boundary conditions. Then decompose $u = \tilde{u} + R$, where $\tilde{u}$ satisfies homogeneous boundary conditions. Give an example of a function R, write the partial differential equation satisfied by $\tilde{u}$, and derive the corresponding weak form.

Ex.3. Provide a variational formulation for each of the following boundary values problems:

(a) $-u'' + u = f, \quad u(0) = 0, u'(1) = 1.$

(b) $-u'' + u' + u = f, \quad u(0) = 0 = u(1).$

(c) $-(\alpha u')' + \beta u' + \gamma u = f \quad u(0) = 0, \quad u'(1) = 3.$

(d) $-(\alpha u')' + \beta u' + \gamma u = f \quad u'(0) = 1, \quad u(1) = 2.$

(e) $-(\alpha u')' + \beta u' + \gamma u = f \quad u(0) = 0, \quad au'(1) + bu(1) = 1$

Ex.4. (From the Strong Problem to the Weak Problem). Let $\Omega \subset \mathbb{R}^N$ be a regular bounded open set. We decompose its boundary $\partial\Omega$ into two distinct non-empty parts Γ_1, Γ_2 with non-zero $(N-1)$-dimensional measure such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$.
Consider the following strong (or boundary value) problems $(\mathscr{P}1), (\mathscr{P}2), (\mathscr{P}3)$:

$$(\mathscr{P}1)\left|\begin{array}{l}\text{Find } u \text{ such that :}\\ \begin{aligned} -\Delta u &= f \quad \text{in } \Omega,\\ u &= g \quad \text{on } \Gamma_1,\\ \partial_n u &= 0 \quad \text{on } \Gamma_2.\end{aligned}\end{array}\right. \qquad (\mathscr{P}2)\left|\begin{array}{l}\text{Find } u \text{ such that :}\\ \begin{aligned} -\operatorname{div}(k\nabla u) + \beta\partial_1 u &= f \quad \text{in } \Omega,\\ u &= g \quad \text{on } \Gamma_1,\\ k\partial_n u &= h \quad \text{on } \Gamma_2.\end{aligned}\end{array}\right. \qquad (\mathscr{P}3)\left|\begin{array}{l}\text{Find } u \text{ such that :}\\ \begin{aligned} -\operatorname{div}(k\nabla u) + \beta u &= f \quad \text{in } \Omega,\\ u &= g \quad \text{on } \Gamma_1,\\ k\partial_n u &= h \quad \text{on } \Gamma_2.\end{aligned}\end{array}\right.$$

Here, f belongs to $L^2(\Omega)$, g, h are two functions assumed to be continuous, and k denotes an element of $L^\infty(\Omega, \mathbb{R}^+)$ satisfying $k(x) \geqslant C > 0$ almost everywhere on Ω. Finally, $\beta > 0$ is a constant.
Give a variational formulation of $(\mathscr{P}1), (\mathscr{P}2), (\mathscr{P}3)$ in the form:

$$(\mathscr{P}_{\mathrm{v}})\left|\begin{array}{l}\text{Find } u \in V \text{ such that}\\ a(u,v) = \ell(v), \quad \forall v \in W.\end{array}\right. \tag{6}$$

In each case, specify the choice of spaces V, W as well as the expressions of the forms $a(\cdot,\cdot)$, $\ell(\cdot)$. When $a(\cdot,\cdot)$ is symmetric, give the definition of the energy whose minimization leads to the study of the problem $\mathscr{P}_{\mathrm{v}}$.

Ex.5. (From the Weak Problem to the Strong Problem). Let $\Omega \subset \mathbb{R}^N$ be a regular bounded open set. We decompose its boundary $\partial\Omega$ into two distinct non-empty parts Γ_1, Γ_2 with non-zero $(N-1)$-dimensional measure such that $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Consider the following weak (or variational) problems:

$$(\mathscr{P}_{\mathrm{v}1})\left|\begin{array}{l}\text{Find } u \in H^1_0(\Omega) \text{ such that :}\\ \displaystyle\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \beta u(x)v(x)\,dx = \int_\Omega f(x)v(x)\,dx, \quad \forall v \in H^1_0(\Omega).\end{array}\right.$$

$$(\mathscr{P}_{\mathrm{v}2})\left|\begin{array}{l}\text{Find } u \in \{\varphi \in H^1(\Omega) \mid \gamma_1(\varphi) = g\} \text{ such that :}\\ \displaystyle\int_\Omega k(x)\nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \beta u(x)\frac{\partial v(x)}{\partial x_1}\,dx + \int_{\Gamma_2} uv\,ds = \int_\Omega f(x)v(x)\,dx, \quad \forall v \in H^1_{0,\Gamma_1}(\Omega)\end{array}\right.$$

Here, $\gamma_1 : H^1(\Omega) \to L^2(\Gamma_1)$ denotes the trace operator on Γ_1, and

$$H^1_{0,\Gamma_1}(\Omega) := \{\varphi \in H^1(\Omega) \mid \gamma_1(\varphi) = 0\}.$$

Provide the strong (partial differential equation and boundary conditions) problems associated with the variational problems $(\mathscr{P}_{\mathrm{v}1})$ and $(\mathscr{P}_{\mathrm{v}2})$.

Ex.6. Consider the function:

$$u(x) = \begin{cases} 1 & \text{if } x > 0\\ 0 & \text{if } x < 0\end{cases}$$

Why the function u is not in $W^{1,p}(-1,1)$ for any $1 \leqslant p \leqslant \infty$?

Ex.7. We recall that $f_+ = \max(0, f(x))$.

(a) Draw the graph of the function ϕ defined by $\phi(x) = (1 - |x|)_+$ for $x \in [-2, 2]$.

(b) Is it true that $\phi \in C[-2,2] \cap C^1(-2,2)$?

(c) Calculate the first (weak) derivative $\phi' = D\phi$ of ϕ on the interval $[-2, 2]$.

(d) Verify that $\phi, \phi' \in L_p(-2,2)$ for all $p \in [1, \infty]$. Hence deduce that $\phi \in W_p^1(-2,2)$ for all $p \in [1, \infty]$.

Ex.8. Suppose that $u(x) = x^\alpha, x \in [0,1]$, where α is a fixed real number, $0 < \alpha < 1$. Show that $u \in C^\infty(0,1)$, but $u \notin W_p^1(0,1)$ for $p \geqslant (1-\alpha)^{-1}$.

Ex.9. Let $\Omega = \left\{(x,y) \in \mathbf{R}^2 : x^2 + y^2 < \frac{1}{4}\right\}$ and consider the function u defined on Ω by $u(x,y) = \log\left|\log\sqrt{x^2+y^2}\right|$. Show that $u \in W_2^1(\Omega)\,(= H^1(\Omega))$ but $u \notin C(\Omega)$.

Ex.10. Sobolev Space H^1 and Continuity.

(a) Let I be a bounded or unbounded interval. Show that every element $u \in H^1(I)$ is continuous and uniformly bounded (first consider the case $I = \mathbb{R}$ and use the density of infinitely continuously differentiable functions with compact support $C_0^\infty(\mathbb{R})$ in $H^1(\mathbb{R})$).

This result is not valid in dimension $N \geqslant 2$ as we will see in the following question. Let B denote the open unit ball in $\mathbb{R}^N$.

(b) For $N = 2$, show that the function u defined by $u(x) = |\ln(|x|)|^\alpha$ belongs to $H^1(B)$ for $0 < \alpha < 1/2$ but is not bounded in the neighborhood of the origin.

(c) For $N \geqslant 3$, show that the function u defined by $u(x) = |x|^{-\beta}$ belongs to $H^1(B)$ for $0 < \beta < (N-2)/2$ but is not bounded in the neighborhood of the origin.