Exercises

Ex.1. a) Formulate the following differential equation :

$$-u'' + u' + u = f, \quad u(0) = 0 = u(1).$$

as a variational problem on $H_0^1(0,1)$

b) Show that the bilinear form from this variational problem is coercive and bounded.

c) State and prove Céa's Lemma.

d) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0,1]$ with maximum width h. Let u_h be the solution to the Galerkin approximation of the variational problem using V_h, and let $I_h : H^2(0,1) \to V_h$ be the interpolation operator onto V_h. Assuming $u \in H^2(0,1)$, and the following result,

$$\|u - I_h u\|_{H^1(0,1)} \leqslant h \|u''\|_{L^2(0,1)},$$

show that

$$\|u - u_h\| \leqslant Dh \|u''\|_{L^2(0,1)},$$

and provide a numerical value for D.

Ex.2. Given that (a,b) is an open interval of the real line, let $H_{E_0}^1(a,b) = \{v \in H^1(a,b) : v(a) = 0\}$.
a) By writing

$$v(x) = \int_a^x v'(\xi)\mathrm{d}\xi, \quad a \leqslant x \leqslant b$$

for $v \in H_{E_0}^1(a,b)$, show that the following (Poincaré-Friedrichs) inequality holds for each $v \in H_{E_0}^1(a,b)$:

$$\|v\|_{L_2(a,b)}^2 \leqslant \frac{1}{2}(b-a)^2 |v|_{H^1(a,b)}^2$$

b) By writing

$$[v(x)]^2 = \int_a^x \frac{d}{d\xi}[v(\xi)]^2 \,\mathrm{d}\xi = 2\int_a^x v(\xi)v'(\xi)\mathrm{d}\xi, \quad a \leqslant x \leqslant b$$

for $v \in H_{E_0}^1(a,b)$, show that the following (Agmon's) inequality holds for each $v \in H_{E_0}^1(a,b)$:

$$\max_{x\in[a,b]} |v(x)|^2 \leqslant 2\|v\|_{L_2(a,b)} |v|_{H^1(a,b)}$$

Ex.3. Given that $f \in L_2(0,1)$, state the weak formulation of each of the following boundary value problems:

a) $-u'' + u = f(x)$ for $x \in (0,1), u(0) = 0, u(1) = 0;$

b) $-u'' + u = f(x)$ for $x \in (0,1), u(0) = 0, u'(1) = 0$;

c) $-u'' + u = f(x)$ for $x \in (0,1), u(0) = 0, u(1) + u'(1) = 0$.

Apply the Lax-Milgram lemma to show that each of the three weak formulations has a (corresponding) unique weak solution

Ex.4. Consider the problem

$$\begin{cases} -u'' - k^2 u = f, & \text{in } (0,\pi) \\ u(0) = 0 = u(\pi) \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ and $k^2 \in \mathbb{R}$.

a) Cast the problem in variational form, stating carefully the spaces employed.

b) For what values of k is the problem not well-posed? (Hint: take $f = 0$ and look for nonzero solutions.)

c) For small values of k, the problem is coercive. For large values of k, the problem is not coercive. For what value of k does the problem lose coercivity?

Ex.5. Let V be the function space defined on $[0,1]$ by

$$V = \left\{ u \in L_2 : \int_0^1 u^2 + (u')^2 \ \mathrm{d}x < \infty \right\}.$$

Consider the variational problem,

$$\text{Find } u \in V \text{ such that } \int_0^1 uv + u'v' \mathrm{d}x = \int_0^1 fv \ \mathrm{d}x, \quad \forall v \in V. \tag{2}$$

Let $0 < x_1 < x_2 < \ldots < x_{n-1} < 1$ define a subdivision of the interval $[0,1]$. Let V_h be a finite dimensional subspace of V, consisting of all functions that are linear in each subinterval, and continuous between subintervals.

a) Formulate the finite element approximation for Equation (2) using V_h, and show how it results in a matrix-vector system of the form

$$K\mathbf{u} = \mathbf{F}.$$

[You do not need to compute the entries of K and $\mathbf{F}$, just provide a general formula for how they are calculated]

b) For the finite element approximation to Equation (2) given above, show that

$$\sum_{ij} K_{ij} = 1.$$

Ex.6. Let us consider the following differential equation

$$-u'' + u' + u = f, \text{ on } [0,1], \quad u(0) = u(1) = 0.$$

a) Formulate its variational problem on $V = H_0^1[0,1]$.

b) Show that variational problem is well posed.

c) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0,1]$ into elements with maximum width h. Let u_h be the solution to the the Ritz-Galerkin approximation of Equation (2) using V_h. Assuming the following result,

$$\min_{v \in V_h} \|u - v\|_{H^1_{[0,1]}} \leqslant h|u|_{H^2_{[0,1]}}$$

for $\gamma > 0$, show that

$$\|u - u_h\|_{H^1_{[0,1]}} \leqslant Dh|u|_{H^2_{[0,1]}},$$

and provide a numerical value for D.

d) Consider the modified variational problem with boundary conditions:

$$u'(0) = \alpha, u'(1) = \beta.$$

Show that this variational problem is well posed.

Ex.7. We consider the following boundary value problem in one dimension.

$$-u'' + (2 + \sin(x))u = f(x), \quad u(0) = 0, u'(1) = 1.$$

a) Construct a formulation of this problem describing a weak solution u in $H^1([0,1])$.

b) Show that the corresponding bilinear form is continuous and coercive in $H^1([0,1])$, and compute the continuity and coercivity constants.

c) What is the required property of f for a unique solution u to exist?

d) Describe the piecewise linear C^0 finite element discretisation of this equation with mesh vertices $[x_0 = 0, x_1, x_2, \ldots, x_n, x_{n+1} = 1]$.

e) Given an arbitrary basis of the finite element space V_h, show that the resulting matrix A is symmetric $\left(A^T = A\right)$ and positive definite, i.e. $x^T A x > 0$ for all x with $x \neq 0$.

f) Show that the numerical solution u_h satisfies $\|u - u_h\|_{H^1([0,1])} = \mathcal{O}(h)$ as $h \to 0$. [You may quote any properties of the interpolation operator $\mathcal{I}_h$ without proof, but must show the other steps.]

Ex.8. Consider the variational problem of finding $u \in H^1([0,1])$ such that

$$\int_0^1 vu + v'u'\mathrm{d}x = \int_0^1 vx \ \mathrm{d}x + v(1) - v(0), \quad \forall v \in H^1([0,1]).$$

After dividing the interval $]0,1[$ into N equispaced cells and forming a $P1$-C^0 finite element space V_N, the error $\|u - u_h\|_{H^1} = 0$ for any $N > 0$. Explain why this is expected.

Ex.9. Let $\overset{\circ}{H}{}^1(]0,1[)$ be the subspace of $H^1(]0,1[)$ such that $u(0) = 0$.

Consider the variational problem of finding $u \in \overset{\circ}{H}{}^1(]0,1[)$ with

$$\int_0^1 u'v'dx = \int_0^{1/2} vdx, \quad \forall v \in \overset{\circ}{H}{}^1(]0,1[).$$

The interval $[0,1]$ is divided into $3N$ equispaced cells (where N is a positive integer). We consider the $\mathbb{P}_1 - C^0$ conforming finite element approximation.

i) Could you apply the Céa lemma and the interpolation error to show that the error $\|u - u_h\|_{H^1}$ converges to zero as $N \to \infty$.

ii) We will see in the next year (EEP: a posteriori error estimate) that:

$$\|u - u_h\|_{H^1} \leqslant c_1 h \|f\|_0 + c_2 \|f - f_h\|_0$$

where f_h is the projection of f in the space of piecewise constant functions $\mathbb{P}_0$. Does this explain why the error $\|u - u_h\|_{H^1}$ is found not to converge to zero as $N \to \infty$.?