Exercises

Ex.1. (Interpolation Error and Convergence in 1D).

For a fixed integer $m > 0$ and $h = 1/N$, we consider the spaces (Lagrange finite elements P_m):

$$V_{mh} := \left\{ f \in C^0([0,1]) \mid f|_{[jh,(j+1)h]} \in P_m, j = 0, \dots, N-1 \right\}$$

where P_m is the set of polynomials of degree less than or equal to m.

(a) Recall why $V_{mh} \subset H^1(0,1)$. What is the dimension of V_{mh}? Exhibit a basis for this space and an interpolation operator r_h from H^1 onto V_{mh}.

(b) Show that for $1 \leqslant n \leqslant m+1$, there exists a constant C such that for any $u \in H^n(]0,1[)$:

$$\|r_1 u - u\|_{H^1} \leqslant C \|u^{(n)}\|_{L^2}.$$

Hints:

- Let R_{n-1} be the L^2 orthogonal projection onto the set of polynomials of degree at most $n-1$. Show that for any $u \in H^n(]0,1[)$ and $n \leqslant m+1$, we have:

$$u - r_1 u = v - r_1 v,$$

  where $v = u - R_{n-1} u$.
- Deduce that it is sufficient to prove the inequality for $v \in \widetilde{H}^n(]0,1[)$, where $\widetilde{H}^n(]0,1[)$ denotes the set of functions in $H^n(]0,1[)$ that are L^2-orthogonal to the set of polynomials of degree $\leqslant n-1$.
- By using induction and a proof by contradiction, show that for $0 \leqslant k < n$, there exists a constant $C > 0$ such that:

$$\|v^{(k)}\|_{L^2} \leqslant C \|v^{(k+1)}\|_{L^2}, \quad \forall v \in \widetilde{H}^n(]0,1[).$$

- Conclude!

(c) Deduce that for $1 \leqslant n \leqslant m+1$, there exists a constant C such that for any $u \in H^n(]0,1[)$:

$$\|r_h u - u\|_{L^2} \leqslant C h^n \|u^{(n)}\|_{L^2}.$$

Similarly, show that there exists a constant C such that for any $u \in H^n(]0,1[)$:

$$\|(r_h u)' - u'\|_{L^2} \leqslant C h^{n-1} \|u^{(n)}\|_{L^2}.$$

(d) Deduce an error estimate for the Lagrange finite element method P_m applied to the problem:

$$-u'' = f \quad \text{on }]0,1[, \quad u(0) = u(1) = 0,$$

when $f \in H^n(0,1)$, $n \geqslant 0$ (distinguish the cases $n \leqslant m-1$ and $n \geqslant m$). Is it beneficial to use $m > 1$ if f is only in L^2?

Ex.2. (Aubin-Nitsche Lemma)

(a) **1D Case:** Consider $u \in H_0^1(0,1)$, the solution to the problem:

$$-u'' = f \quad \text{on }]0,1[,$$

and its discretization u_h using Lagrange finite elements P_1. Show that we have the estimate:

$$\|u - u_h\|_{L^2(0,1)} \leqslant Ch^2 \|f\|_{L^2(0,1)}.$$

Hint: Consider the auxiliary problem: $w \in H_0^1(0,1)$

$$-w'' = u - u_h \quad \text{on }]0,1[,$$

and use w and its approximation $r_h w$ in V_h to estimate $\|u - u_h\|_{L^2}^2$.

(b) **A Generalization:** Consider $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^N$, as the solution to the variational problem $u \in H_0^1(\Omega)$:

$$a(u,v) = \int_\Omega fv\,dx \quad \forall v \in H_0^1(\Omega),$$

where a is a continuous and coercive bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$. Suppose that for every $f \in L^2(\Omega)$, the solution to this variational problem satisfies $u \in H^2(\Omega)$ and there exists a constant $C > 0$, independent of f, such that:

$$\|u\|_{H^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.$$

Suppose this property also holds for the adjoint problem, $w \in H_0^1(\Omega)$:

$$a(v,w) = \int_\Omega fv\,dx \quad \forall v \in H_0^1(\Omega).$$

Show that the result from question 1 generalizes to this case.

Ex.3. (Lower bound). Suppose that u is smooth. Show that:

$$\inf_{v_h \in V_h^1} \|u - v_h\|_1 \geqslant Ch = CN^{-\frac{1}{d}} \tag{1}$$

Ex.4. Consider the problem

$$-Tu'' = -\rho g, \quad u(0) = 0 = u(L),$$

with $L = 10$ m, $\rho = 1$ kg/m, $g = 9.8$ ms^{-2}, and $T = 98$ N. Here u is the vertical deflection of a hanging cable sagging under gravity. Compute the Galerkin approximation to this problem using the trial space

$$V_h = \operatorname{span}\left\{\sin\left(\frac{\pi x}{L}\right), 1 - \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right)\right\}.$$

Note that each basis function of V_h satisfies the boundary conditions, and hence $u_h \in V_h$ will also. (Note: it may be more convenient to use a symbolic algebra package such as sympy, maple or mathematica for this.)

Ex.5. (From Riesz to Lax-Milgram). Let $\Omega \subset \mathbb{R}^N$ be a bounded regular open set and $\mathbf{b} \in C^1(\Omega)^N$ a given divergence-free vector field. We are interested in the problem

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ -\Delta u + (\mathbf{b} \cdot \nabla u) = f \text{ in } \Omega, \\ \qquad\qquad\qquad u = 0 \text{ on } \partial\Omega \end{cases} \tag{2}$$

where f denotes a source term belonging to $L^2(\Omega)$.

(a) Write the variational formulation (VF) naturally associated with this problem in a specified space. Show that if u is a solution of (VF) belonging to $H^2(\Omega)$, then u satisfies (2).

(b) Can the Riesz representation theorem be used to show that (VF) has a unique solution? What other tool can be employed? Prove that the mapping

$$A : L^2(\Omega) \to H^1(\Omega), \quad f \mapsto u,$$

where u is the solution of (VF), is linear and continuous.

(c) Would we obtain an identical result if the divergence of $\mathbf{b}$ had strictly positive values?

(d) Let W be a finite-dimensional subspace of the space introduced in question 1. Show that there exists a unique solution to the problem

$$\begin{cases} \text{Find } u \in W \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \mathbf{b}(x) \cdot \nabla u v \, dx = \int_\Omega f v \, dx, \quad \forall v \in W. \end{cases} \tag{3}$$

Ex.6. (Laplacian with Neumann Boundary Conditions).

Consider $\Omega \subset \mathbb{R}^N$ a regular bounded connected open set. We are interested in the Laplace problem with Neumann boundary conditions:

$$(P_N) \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that } : \\ \quad -\Delta u = f \quad \text{in } \Omega, \\ \quad \dfrac{\partial u}{\partial n} = g \quad \text{on } \partial\Omega, \end{cases} \tag{4}$$

where f and g represent source terms belonging respectively to $L^2(\Omega)$ and $L^2(\partial\Omega)$.

(a) Can we hope to prove a uniqueness result for the solution to problem (P_N)? Why?

(b) Provide a necessary condition (known as the compatibility condition) that f and g must satisfy for (P_N) to admit a solution. Hint: You may integrate the PDE over Ω.

(c) Define the space:

$$H^1_{\#}(\Omega) := \left\{ \varphi \in H^1(\Omega) \mid \int_\Omega \varphi(x)\, dx = 0 \right\}.$$

Show that if u satisfies problem (P_N), then there exists a constant α such that $w = u - \alpha$ is a solution of the problem:

$$(P_{N_v})\text{Find } w \in H^1_{\#}(\Omega) \text{ such that } a(w, v) = \ell(v), \quad \forall v \in H^1_{\#}(\Omega),$$

where the expressions for $a(\cdot, \cdot)$ and $\ell(\cdot)$ are to be given. Conversely, prove that if w is a solution of (P_{N_v}) belonging to $H^2(\Omega)$, then w satisfies (P_N) when f and g satisfy the compatibility condition obtained in question 2.

(d) Show that $H^1_{\#}(\Omega)$ equipped with the inner product of $H^1(\Omega)$ is a Hilbert space.

(e) Show that $H^1_{\#}(\Omega)$ equipped with the inner product $(\varphi, \varphi') \mapsto \int_\Omega \nabla\varphi(x) \cdot \nabla\varphi'(x)\, dx$ is also a Hilbert space.

(f) Show that (P_{N_v}) admits a unique solution. Is it necessary to assume that f, g satisfy the compatibility condition from question 2? Establish that the application:

$$A : L^2(\Omega) \times L^2(\partial\Omega) \to H^1_{\#}(\Omega),$$
$$(f, g) \mapsto w,$$

where w is the solution of (P_{N_v}), is linear and continuous.