<u>Exercises</u>

Ex.1. Let $f \in L^2(0,1)$. We consider the following mixed problem:

$$\begin{cases} \text{find } (\sigma, u) \in H^1(\Omega) \times L^2(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \sigma\tau \mathrm{d}x - \int_\Omega u\tau' \mathrm{d}x = 0,\ \forall \tau \in H^1(\Omega) \\ \displaystyle\qquad\qquad -\int_\Omega \sigma' v \ \mathrm{d}x = -\int_\Omega fv \ \mathrm{d}x,\ \forall v \in L^2(\Omega) \end{cases} \tag{M}$$

(a) Shaw that then the kernel

$$\ker b := \left\{ \tau \in H^1(\Omega) : \int_\Omega \tau' v \ \mathrm{d}x = 0 \text{ for all } v \in L^2(\Omega) \right\}$$

contains only the constant functions.

(b) Shaw that the problem (M) admits a unique solution.

(c) For the discrete problem (M), shaw that the discretisation:

i) $\mathbb{P}_1 \times \mathbb{P}_1$ is ill-posed.

ii) $\mathbb{P}_1 \times \mathbb{P}_0$ is well posed.

Ex.2. (a) Let $V = (H^1(\Omega))^2$ and $Q = L_0^2(\Omega)$, where Ω is a convex polygonal domain. Let $b : V \times Q \to \mathbb{R}$ be the bilinear form

$$b(v, q) = \int_\Omega q \nabla \cdot v \ \mathrm{d}x$$

Assuming the result that the divergence is surjective from V to Q, show that b satisfies the inf-sup condition

$$\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geqslant \beta$$

for some constant β.

(b) Define the operator $\delta : Q \to V$ by

$$\int_\Omega w \cdot \delta q \ \mathrm{d}x = b(w, q), \quad \forall w \in V$$

Show that the kernel of $\delta, \mathrm{Ker}(\delta)$, is empty. (Hint: find a suitable choice of test function.)

(c) Let $V_h \subset V$ and $Q_h \subset Q$ be finite element spaces chosen for the discretisation of Stokes' equation. Let $\Pi_h : V \to V_h$ satisfy condition (FC) of Fortin's trick, i.e.

$$b\left(v - \Pi_h v, q\right) = 0, \quad \forall v \in V, q \in Q_h \tag{FC}$$

Define the discrete operator $\delta_h : Q_h \to V_h$ by

$$\int_\Omega w \cdot \delta_h q \, \mathrm{d}x = b(w, q), \quad \forall w \in V_h$$

Show that $\mathrm{Ker}\,(\delta_h) \subseteq \mathrm{Ker}(\delta)$.

(d) Consider a mesh consisting of squares subdividing into right angle triangles by joining the top left vertex and the bottom right vertex of each square, and consider the discretisation for Stokes with continuous linear Lagrange elements for each component of the velocity and continuous linear Lagrange elements for the pressure.

By considering the function $p \in Q_h$ taking vertex values in an alternating pattern as indicated in Figure 2, show that $\ker(\delta_h) \subset \ker(\delta)$ in this case.
Do V_h and Q_j satisfy condition 1 of Fortin's trick?