AN
MASTER

2024-2025
Kasdi Merbah University

Exam.

1 QMC. Find the correct answer (8 pts).

Q1. For the function $f(x) = x^2$. The Bernstein polynomial of degree n is :

a) $nx^2 + x + 1$.
b) $\frac{n-1}{n}x^2 + \frac{x}{n}$.
c) $x^n + 2x - 1$.

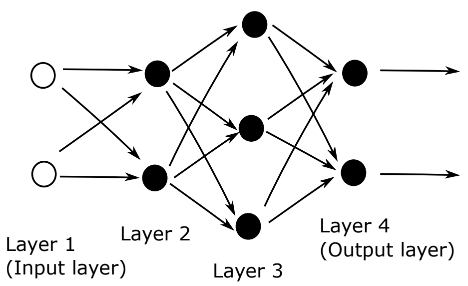
Q2. For $\rho > 0$, consider $v_\rho(x) = x^\rho, \quad x \in [a, b], \quad 0 \leq a < b \leq 1$ then

a) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_\infty = \frac{v_\rho(b)-v_\rho(a)}{2}$.
b) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_\infty = \|v_\rho - \frac{v_\rho(b)+v_\rho(a)}{2}\|_\infty$.
c) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_\infty = \frac{1}{2}\int_a^b |v'_\rho| dx$.

Q3. For the L^2-norm

a) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_{L^2(a,b)} = \frac{v_\rho(b)-v_\rho(a)}{2}$.
b) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_{L^2(a,b)} = \|v_\rho - \frac{v_\rho(b)+v_\rho(a)}{2}\|_{L^2(a,b)}$.
c) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_{L^2(a,b)} = \|v_\rho - \frac{1}{b-a}\int_a^b v_\rho dx\|_{L^2(a,b)}$.

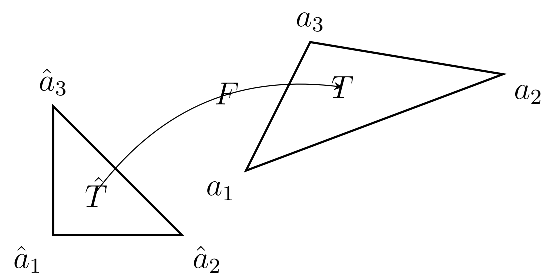
Q4. The number of parameters (weights w_{ij} and biases b_j) for the following deep neural network
is :



a) 25.
b) 21.
c) 23.

2 Problem : Quadratures in 2D.(12 pts).

Let $\hat{T}$ be the reference triangle defined by the vertices $\hat{a}_1 = (0,0)$, $\hat{a}_2 = (1,0)$ and $\hat{a}_3 = (0,1)$ and let T be an arbitrary non-degenerate triangle defined by the vertices $a_1 = (x_1, y_1), a_2 = (x_2, y_2)$ and $a_3 = (x_3, y_3)$.



1. Compute the integrals of $\hat{x}$ and $\hat{y}$ over $\hat{T}$: $\iint_{\hat{T}} \hat{x}\, d\hat{x}\, d\hat{y}$ and $\iint_{\hat{T}} \hat{y}\, d\hat{x}\, d\hat{y}$.
2. Show that the integrals of x and y over an arbitrary triangle T are :
$$\iint_T x\, dx\, dy = \bar{x}|T| \quad \text{and} \quad \iint_T y\, dx\, dy = \bar{y}|T|.$$
where $(\bar{x}, \bar{y})$ is the centroid of T : $\bar{x} = \frac{1}{3}\sum_{m=1}^{3} x_m, \quad \bar{y} = \frac{1}{3}\sum_{m=1}^{3} y_m$.
3. For a continuous function f we define the following quadrature formula[1].
$$\iint_T f(x,y)dx\, dy \approx Q(f) = \frac{|T|}{3}\sum_{m=1}^{3} f(a_m) \tag{1}$$
Let p be a linear function in 2D, i.e, $p(x,y) = c_1 x + c_2 y + c_3$, where $c_1, c_2, c_3 \in \mathbb{R}$. Show that the quadrature formula (1) is <u>exact</u> for linear functions on the triangle T.
4. Compute the integral $\iint_T (x + y + 1)\, dx\, dy$, in terms of $|T|$ and $\{(x_m, y_m)\}_{1 \le m \le 3}$.

3 Bonus : (2 pts).

The Clenshaw–Curtis formula consists of integrating the degree $n-1$ polynomial interpolant through n Chebyshev points
$$x_j = \cos(j\pi/(n-1)), \quad 0 \le j \le n-1. \tag{2}$$
Write a chebfun-Matlab code that computes the integral
$$\int_{-1}^{1} \exp(-1/x^2)dx$$
using the Clenshaw–Curtis formula.

1. where $|T|$ is the area of T i.e, $|T| = \iint_T dx\, dy$

Answer.

1 QMC. Find the correct answer (4 pts) and justify your answer (4 pts).

Q1. For the function $f(x) = x^2$. The Bernstein polynomial of degree n is :

a) $nx^2 + x + 1$.
b) $\frac{n-1}{n}x^2 + \frac{x}{n}$. ✓
c) $x^n + 2x - 1$.

Q2. For $\rho > 0$, consider $v_\rho(x) = x^\rho, \quad x \in [a, b], \quad 0 \le a < b \le 1$ then

a) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_\infty = \frac{v_\rho(b)-v_\rho(a)}{2}$. ✓
b) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_\infty = \|v_\rho - \frac{v_\rho(b)+v_\rho(a)}{2}\|_\infty$. ✓
c) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_\infty = \frac{1}{2}\int_a^b |v'_\rho| dx$.

Q3. For the L^2-norm

a) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_{L^2(a,b)} = \frac{v_\rho(b)-v_\rho(a)}{2}$.
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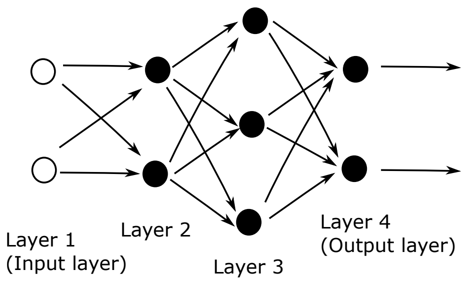
c) $\inf\limits_{c\in\mathbb{R}} \|v_\rho - c\|_{L^2(a,b)} = \|v_\rho - \frac{1}{b-a}\int_a^b v_\rho dx\|_{L^2(a,b)}$. ✓

Q4. The number of parameters (weights w_{ij} and biases b_j) for the following deep neural network



is :

a) 25.
b) 21.
c) 23. ✓

1. Let us compute

$$\iint_{\hat T} \hat x \, d\hat x \, d\hat y = \int_0^1 \int_0^{1-\hat x} \hat x \, d\hat y \, d\hat x = \int_0^1 [\hat x \hat y]_{\hat y=0}^{\hat y=1-\hat x} \, d\hat x = \int_0^1 \hat x(1-\hat x)\, d\hat x$$
$$= (\frac{\hat x^2}{2} - \frac{\hat x^3}{3})\Big|_0 1 = \frac{1}{6} = \frac{|\hat T|}{3}$$

2. The triangle T is related to $\hat T$ by the afine map

$$F\begin{pmatrix} \hat x \\ \hat y \end{pmatrix} = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}\begin{pmatrix} \hat x \\ \hat y \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$|J| = |\nabla F| = |\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}| = |(x_2-x_1)(y_3-y_1) - (x_3-x_1)(y_2-y_1)| = 2|T|$$

$$\begin{aligned}
\iint_T x \, dx \, dy &= \iint_T (1,0)\cdot\begin{pmatrix} x \\ y \end{pmatrix} dx\, dy \\
&= \iint_{\hat T} (1,0)\cdot F\begin{pmatrix} \hat x \\ \hat y \end{pmatrix} |J| \, d\hat x \, d\hat y \\
&= \int_0^1 \int_0^{1-\hat x} (1,0)\cdot\begin{pmatrix} (x_2-x_1)\hat x + (x_3-x_1)\hat y + x_1 \\ (y_2-y_1)\hat x + (y_3-y_1)\hat y + y_1 \end{pmatrix} |J| \, d\hat x \, d\hat y \\
&= 2|T| \left[(x_2-x_1)(\frac{\hat x^2}{2} - \frac{\hat x^3}{3}) - (x_3-x_1)\frac{(1-\hat x)^3}{6} - \frac{x_1(1-\hat x)}{2} \right]_{\hat x=0}^{\hat x=1} \\
&= |T|\left(\frac{x_1+x_2+x_3}{3}\right) = \bar x |T|
\end{aligned}$$

$$\begin{aligned}
\iint_T y \, dx \, dy &= \iint_T (0,1)\cdot\begin{pmatrix} x \\ y \end{pmatrix} dx\, dy \\
&= \iint_{\hat T} (1,0)\cdot F\begin{pmatrix} \hat x \\ \hat y \end{pmatrix} |J| \, d\hat x \, d\hat y \\
&= \int_0^1 \int_0^{1-\hat y} (0,1)\cdot\begin{pmatrix} (x_2-x_1)\hat y + (x_3-x_1)\hat y + x_1 \\ (y_2-y_1)\hat x + (y_3-y_1)\hat y + y_1 \end{pmatrix} |J| \, d\hat x \, d\hat y \\
&= 2|T| \left[(y_2-y_1)(\frac{\hat y^2}{2} - \frac{\hat y^3}{3}) - (y_3-y_1)\frac{(1-\hat y)^3}{6} - \frac{y_1(1-\hat y)}{2} \right]_{\hat y=0}^{\hat y=1} \\
&= |T|\left(\frac{y_1+y_2+y_3}{3}\right) = \bar y |T|
\end{aligned}$$

3. If $f(x,y) = c_1x + c_2y + c_3$ and $G(\bar{x},\bar{y})$ is the centroid or the barycenter. From the previous question we have

$$\begin{aligned}\iint_T f(x,y)\ dx\,dy &= c_1\bar{x}|T| + c_2\bar{y}|T| + c_3|T| \\ &= |T|f(G) \qquad (3)\end{aligned}$$

On the other hand we have :

$$\begin{aligned}Q(f) &= \frac{|T|}{3}\sum_{m=1}^{3} f(a_m) \\ &= \frac{|T|}{3}\sum_{m=1}^{3}\left(c_1x_m + c_2y_m + c_3\right). \\ &= \frac{|T|}{3}\left(c_1\sum_{m=1}^{3} x_m + c_2\sum_{m=1}^{3} y_m + 3c_3\right) \\ &= |T|\left(c_1\sum_{m=1}^{3}\frac{x_m}{3} + c_2\sum_{m=1}^{3}\frac{y_m}{3}\right) + c_3|T| \\ &= |T|f(G).\end{aligned}$$

Comparing this with the integral (3) we see that

$$Q(f) = \iint_T f(x,y)\ dx\,dy$$

Hence, the formula is exact for linear functions.

4. We can compute the $\iint_T (x+y+1)\,dx\,dy$ using the quadratrue since

$$\begin{aligned}\iint_T (x+y+1)\,dx\,dy = \iint_T (x+y+1)\,dx\,dy &= \frac{|T|}{3}(x_1+y_1+1+x_2+y_2+1+x_3+y_3+1) \\ &= |T|(\bar{x}+\bar{y}+1)\end{aligned}$$