Master Maths
First year

2022-2023
Kasdi Merbah University

Homework 1.

Exercise 1

i) Prove or disprove : if $f \in C([-1,1])$ is odd then a best approximation to f by odd polynomials of degree at most n is a best approximation to f among all polynomials of degree n.

ii) Prove or disprove : if $f \in C([-1,1])$ has mean value zero, then a best approximation to f by polynomials of degree at most n with mean value zero is a best approximation to f among all polynomials of degree n.

Exercise 2

State and prove the Jackson theorem in $C^k([a,b])$ paying attention to the dependence of the constant on the interval $[a,b]$. (You can use the Jackson theorems proved in class.)

Exercise 3

If $f \in C(\mathbb{R})$ and $\delta > 0$ define $R_\delta f \in C(\mathbb{R})$ by

$$R_\delta f(x) = \frac{1}{\delta}\int_{x-\delta/2}^{x+\delta/2} f(t)dt.$$

Note that $R_\delta f \in C^1(\mathbb{R})$.

i) Prove that $\|f - R_\delta f\| \leq \omega(\delta)$, where ω denotes the modulus of continuity of f (i.e., $\omega(\delta)$ is the supremum of $|f(x) - f(y)|$ over x, y for which $|x - y| \leq \delta$).

Exercise 4

1. Let $f \in C_{2\pi}$ and let ω denote its modulus of continuity. Using the Jackson theorem in $C^1_{2\pi}$ and the regularization operator of the previous problem, prove that

$$\inf_{p\in\mathcal{T}_n} \|f - p\|_\infty \leq c\omega\left(\frac{1}{n+1}\right).$$

Give an explicit expression for c.

Exercise 5

Let $f \in C([-1,1])$ and let ω denote its modulus of continuity.

1. Prove that

$$\inf_{p\in\mathcal{P}_n} \|f - p\|_\infty \leq c\omega\left(\frac{1}{n+1}\right).$$

Give an explicit expression for c.

2. Suppose that $f \in C([-1,1])$ satisfies the Holder condition $|f(x)-f(y)| \leq M|x-y|^\alpha$ where $M, \alpha > 0$. What can you say about the rate of convergence of the best uniform approximation to f by polynomials of increasing degree ?

Solutions.