Master Maths
First year

2022-2023
Kasdi Merbah University

Homework 3

Exercise 1

Construct orthogonal polynomials of degrees 0, 1 and 2 on the interval $(0, 1)$ with the weight function $w(x) = -\ln x$.

Exercise 2

Let the polynomials

$$\phi_j, \quad j = 0, 1, \ldots,$$

form an orthogonal system on the interval $(-1, 1)$ with respect to the weight function $w(x) = 1$.

- Show that the polynomials

$$\phi_j((2x - a - b)/(b - a)), \quad j = 0, 1, \ldots,$$

  represent an orthogonal system for the interval (a, b) and the same weight function.

Exercise 3

Suppose that the polynomials $\phi_j, \quad j = 0, 1, \ldots,$ form an orthogonal system on the interval $(0, 1)$ with respect to the weight function $w(x) = x^\alpha, \quad \alpha > 0$.

- Find, in terms of ϕ_j, a system of orthogonal polynomials for the interval $(0, b)$ and the same weight function.

Exercise 4

1. Show that, for $0 \leq k \leq n$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left(1 - x^2\right)^n = \left(1 - x^2\right)^{n-k} q_k(x),$$

where q_k is a polynomial of degree k.

2. Deduce that all the derivatives of the function $(1 - x^2)^n$ of order less than n vanish at $x = \pm 1$.

3. Define $\varphi_j(x) = (\mathrm{d}/\mathrm{d}x)^j\,(1 - x^2)^j$, and show by repeated integration by parts that

$$\int_{-1}^{1} \varphi_k(x)\varphi_j(x)\mathrm{d}x = 0, \quad 0 \leq k < j.$$

Exercise 5

1. Show, that, for $0 \le k \le j$,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k x^j \mathrm{e}^{-x} = x^{j-k} q_k(x) \mathrm{e}^{-x}$$

where $q_k(x)$ is a polynomial of degree k.

2. The function φ_j is defined for $j \ge 0$ by

$$\varphi_j(x) = \mathrm{e}^x \frac{\mathrm{d}^j}{\mathrm{d}x^j}\left(x^j \mathrm{e}^{-x}\right).$$

➩ Show that, for each $j \ge 0, \varphi_j$ is a polynomial of degree j, and that these polynomials form an orthogonal system on the interval $(0, \infty)$ with respect to the weight function $w(x) = \mathrm{e}^{-x}$.

3. Write down the polynomials with $j = 0, 1, 2$ and 3.

4. Suppose that $\varphi_j, j = 0, 1, \ldots$, form a system of orthogonal polynomials with weight function $w(x)$ on the interval (a, b).

5. Show that, for some value of the constant $C_j, \varphi_{j+1}(x) - C_j x \varphi_j(x)$ is a polynomial of degree j, and hence that

$$\varphi_{j+1}(x) - C_j x \varphi_j(x) = \sum_{k=0}^{j} \alpha_{jk} \varphi_k(x), \quad \alpha_{jk} \in \mathbb{R}.$$

Use the orthogonality properties to show that $\alpha_{jk} = 0$ for $k < j - 1$, and deduce that the polynomials satisfy a recurrence relation of the form

$$\varphi_{j+1}(x) - (C_j x + D_j)\, \varphi_j(x) + E_j \varphi_{j-1}(x) = 0, \quad j \ge 1.$$

Exercise 6

In the notation of the previous Exercise suppose that the normalisation of the polynomials is so chosen that for each j the coefficient of x^j in $\varphi_j(x)$ is positive. Show that $C_j > 0$ for all j. By considering

$$\int_a^b w(x)\varphi_j(x)\left[\varphi_j(x) - C_{j-1} x \varphi_{j-1}(x)\right] \mathrm{d}x$$

➩ Show that

$$\int_a^b w(x) x \varphi_{j-1}(x) \varphi_j(x) \mathrm{d}x > 0,$$

and deduce that $E_j > 0$ for all j. Hence show that for all positive values of j the zeros of φ_j and φ_{j-1} interlace.

Exercise 7

1. Using the weight function w on the interval (a, b) to find the best polynomial approximation p_n of degree n in the 2 -norm to the function x^{n+1}.
2. Show that

$$\left\|x^{n+1} - p_n\right\|_2^2 = \int_a^b w(x)\varphi_{n+1}^2 \, \mathrm{d}x / \left[c_{n+1}^{n+1}\right]^2,$$

   where c_{n+1}^{n+1} is the coefficient of x^{n+1} in $\varphi_{n+1}(x)$.
3. Write down the best polynomial approximation of degree 2 to the function x^3 in the 2 -norm with $w(x) \equiv 1$ on the interval $(-1, 1)$, and evaluate the 2 -norm of the error.

Exercise 8

Suppose that the weight w is an even function on the interval $(-a, a)$, and that a system of orthogonal polynomials $\varphi_j, j = 0, \ldots, n$, on the interval $(-a, a)$ is constructed by the Gram-Schmidt process.

1. Show that, if j is even, then φ_j is an even function, and that, if j is odd, then φ_j is an odd function. Now suppose that the best polynomial approximation of degree n in the 2 -norm to the function f on the interval $(-a, a)$ is expressed in the form

$$p_n(x) = \gamma_0\varphi_0(x) + \cdots + \gamma_n\varphi_n(x).$$

2. Show that if f is an even function, then all the odd coefficients γ_{2j-1} are zero, and that if f is an odd function, then all the even coefficients γ_{2j} are zero.

Exercise 9

The function $H(x)$ is defined by $H(x) = 1$ if $x > 0$, and $H(-x) = -H(x)$. Construct the best polynomial approximations of degrees 0,1 and 2 in the 2-norm to this function over the interval $(-1, 1)$ with weight function $w(x) \equiv 1$. (It may not appear very useful to consider a polynomial approximation to a discontinuous function, but representations of such functions by Fourier series will be useful. Note that the function H belongs to $\mathrm{L}_w^2(-1, 1)$.)

Solutions.

1. On note $w(x) = -\ln(x)$

$$\|p\|_w^2 = (p,p)_w = \int_0^1 p^2(x)w(x)\ dx$$

Observer que :

$$-\int_0^1 x^k \ln(x)\ dx = \frac{1}{(k+1)^2}$$

On cherche p_0, p_1 et p_2,

$$\begin{aligned}
&p_0 = 1 \implies \|p_0\|_w^2 = 1\\
&p_1 = x - \frac{(x,p_0)_w}{\|p_0\|_w^2}p_0 = x - \frac{1}{4} \implies \|p_1\|_w^2 = \frac{7}{144}\\
&p_2 = x^2 - \frac{(x^2,p_1)_w}{\|p_1\|_w^2}p_1 - \frac{(x^2,p_0)_w}{\|p_0\|_w^2}p_0 = x^2 - \frac{5}{7}x + \frac{17}{258}
\end{aligned}$$

2. Il suffit de vérifier que :

$$I_{ij} = \int_a^b \phi_i\left(\frac{2x-a-b}{b-a}\right)\phi_j\left(\frac{2x-a-b}{b-a}\right)\ dx = 0, \quad \text{si } i \neq j$$

On effectue le changement de variable :

$$t = \frac{2x-a-b}{b-a} \implies dt = \frac{2}{b-a}dx$$

alors,

$$I_{ij} = \frac{b-a}{2}\int_{-1}^1 \phi_i(t)\phi_j(t)dt = 0$$

3. Nous avons

$$\int_0^1 \varphi_i(x)\varphi_j(x)x^\alpha\ dx = 0, \quad \text{si } i \neq j$$

On effectue le changement de variable

$$x = \frac{t}{b} \implies x^\alpha dx = \frac{t^\alpha}{b^{\alpha+1}}dt$$

alors,

$$\frac{1}{b^{\alpha+1}} \int_0^b \varphi_i(t)\varphi_j(t)t^\alpha \; dt = 0, \quad \text{si } i \neq j$$

donc si $\{\varphi_j, j = 1, \ldots\}$ est un système orthogonal sur $(0, 1)$ alors, $\{\phi_j(x)\}$ avec

$$\phi_j(x) = \varphi_j(\frac{x}{b}), \quad j = 1, \ldots$$

est un système orthogonal sur $(0, b)$.

4. 1) Par récurrence, supposons

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^k \left(1 - x^2\right)^n = \left(1 - x^2\right)^{n-k} q_k(x),$$

alors,

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{k+1} \left(1 - x^2\right)^n &= \frac{d}{dx}\left[\left(1 - x^2\right)^{n-k} q_k(x)\right] \\
&= -2(n-k)x(1-x^2)^{n-k-1}q_k(x) + \left(1 - x^2\right)^{n-k} q_k'(x) \\
&= (1-x^2)^{n-k-1}\left(-2(n-k)xq_k(x) + (1-x^2)q_k'(x)\right) \\
&= (1-x^2)^{n-k-1}q_{k+1}(x) \\
&= (1-x^2)^{n-(k+1)}q_{k+1}(x)
\end{aligned}$$

2) Donc si $k < n$ la dérivée de $(1 - x^2)^n$ s'annule en ± 1.

3) Par définition :

$$\begin{aligned}
\int_{-1}^1 \varphi_k(x)\varphi_j(x) \; dx &= \int_{-1}^1 D^k(1-x^2)^k D^j(1-x^2)^j \; dx \\
&= (-1)^k \int_{-1}^1 \underbrace{D^{2k}(1-x^2)^k}_{=Cte} D^{j-k-1}(1-x^2)^j \; dx \\
&= 0
\end{aligned}$$