Master Maths
First year

2024-2025
Kasdi Merbah University

Homework 4

Ex. 1. With the usual notation for the Newton-Cotes quadrature formula and using the equally spaced quadrature points :

$$x_k = a + kh \quad \text{for} \quad k = 0, 1, \ldots, n \quad \text{and} \quad n \geq 1,$$

show that $w_k = w_{n-k}$ for $k = 0, 1, \ldots, n$.

Ex. 2. By considering the polynomial

$$\left(x - \frac{a+b}{2}\right)^{n+1}, \quad n \geq 1,$$

show that the Newton-Cotes formula using $n+1$ points $x_k, k = 0, 1, \ldots, n$, is exact for all polynomials of degree $n+1$ whenever n is even.

Ex. 3. A quadrature formula on the interval $[-1, 1]$ uses the quadrature points $x_0 = -\alpha$ and $x_1 = \alpha$, where $0 < \alpha \leq 1$:

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx w_0 f(-\alpha) + w_1 f(\alpha)$$

The formula is required to be exact whenever f is a polynomial of degree 1.

a. Show that $w_0 = w_1 = 1$, independent of the value of α.
b. Show also that there is one particular value of α for which the formula is exact also for all polynomials of degree 2 . Find this α, and show that, for this value, the formula is also exact for all polynomials of degree 3 .

Ex. 4. The Newton-Cotes formula with $n = 3$ on the interval $[-1, 1]$ is

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx w_0 f(-1) + w_1 f(-1/3) + w_2 f(1/3) + w_3 f(1).$$

a. Using the fact that this formula is to be exact for all polynomials of degree 3, to show that

$$2w_0 + 2w_1 = 2 \tag{1}$$
$$2w_0 + \frac{2}{9}w_2 = \frac{2}{3} \tag{2}$$

b. Find the values of the weights w_0, w_1, w_2 and w_3.

Ex. 5. For each of the functions $1, x, x^2, \ldots, x^6$, find the difference between $\int_{-1}^{1} f(x)\mathrm{d}x$ and (i) Simpson's rule, (ii) the formula derived in Exercise 4.
Deduce that for every polynomial of degree 5, formula (ii) is more accurate than formula (i). Find a polynomial of degree 6 for which formula (i) is more accurate than formula (ii).

Ex. 6. Let us consider the integrals :

$$I_4 = \int_0^1 x^4 \,\mathrm{d}x \quad \text{and} \quad I_5 = \int_0^1 x^5 \,\mathrm{d}x$$

a. Write down the errors in the approximation of I_4 and I_5 by :
the trapezium rule and Simpson's rule.

b. Find the value of the constant C for which the trapezium rule gives the correct result for the calculation of

$$\int_0^1 \left(x^5 - Cx^4\right) \mathrm{d}x$$

c. Show that the trapezium rule gives a more accurate result than Simpson's rule when $\frac{15}{14} < C < \frac{85}{74}$.