Lecture 2: Residual a posteriori error estimate

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Contents

General framework

Consider the following problem:

$$\begin{cases} \text{Find } u \in W \text{ such that} \\ a(u,v) = f(v), \quad \forall v \in V \end{cases} \tag{1}$$

where :

- W and V are Hilbert spaces of functions defined over a domain Ω,
- $f \in V'$, and $a \in \mathcal{L}(W \times V; \mathbb{R})$.

We assume that the bilinear form a satisfies the conditions of the Babushka Theorem. Problem (1) is therefore well-posed.

Let W_h and V_h be two approximation spaces constructed from a family $\{\mathcal{T}_h\}_{h>0}$ of meshes of Ω.
Consider the approximate problem:

$$\begin{cases} \text{Find } u_h \in W_h \text{ such that} \\ a_h\left(u_h, v_h\right) = f_h\left(v_h\right), \quad \forall v_h \in V_h \end{cases} \tag{2}$$

where a_h and f_h are approximations to the bilinear form a and the linear form f, respectively. We assume that (2) is well-posed and that suitable consistency and approximability properties hold to ensure that u_h converges to the exact solution u as $h \to 0$.

Definition

A function $e(h, u_h, f)$ is said to be an a posteriori error estimate if

$$\|u - u_h\|_{W(h)} \leq e(h, u_h, f) \tag{3}$$

Furthermore, if $e(h, u_h, f)$ can be localized in the form

$$e(h, u_h, f) = \left(\sum_{K \in \mathcal{T}_h} \eta_K (u_h, f)^2 \right)^{\frac{1}{2}} \tag{4}$$

the quantities (u_h, f) are called local error indicators.

Remark

The estimate (3) is sometimes called a reliability property since it shows that $e(h, u_h, f)$ controls the error $u - u_h$ in the natural stability norm.

Clément interpolant

The Lagrange interpolant is not well defined for the functions of the Sobolev space $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^d, d \geq 2$.

Clément interpolant

The Lagrange interpolant is not well defined for the functions of the Sobolev space $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^d$, $d \geq 2$. An interpolation technique to handle functions in L^1 using H^1-conformal Lagrange finite elements was first analyzed by Clément.

Theorem (Clément interpolant)

There exists an operator $\mathcal{C}_h$ from $H^m(\Omega)$ to $H^m(\Omega)$ such that, for every triangle $T \in \mathcal{T}_h$, every edge $e \in \mathcal{E}_h$, and every function $v \in H^m(\Omega)$, there exists a constant $c > 0$ such that, for $0 \leq m \leq \ell$:

$$\begin{aligned} \|v - \mathcal{C}_h v\|_{m,T} &\leq c \; h_T^{\ell-m} \|v\|_{\ell,V(T)} \\ \|v - \mathcal{C}_h v\|_{m,e} &\leq c \; h_e^{\ell-m-1/2} \|v\|_{\ell,V(e)} \end{aligned} \tag{5}$$

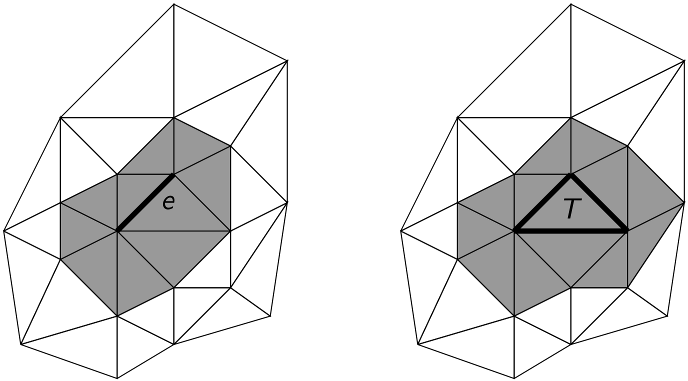

Figure: L'ensemble $V(e)$ et l'ensemble $V(T)$

Let Ω be a polyhedron in $\mathbb{R}^d$, $f \in L^2(\Omega)$, and consider the problem:

$$\begin{cases} \text{Seek } u \in H_0^1(\Omega) \text{ such that} \\ a(u,v) = \int_\Omega fv, \quad \forall v \in H_0^1(\Omega) \end{cases} \tag{6}$$

where $a(u,v) = \int_\Omega \nabla u \cdot \nabla v$. Leaving room for generalizations, we shall not use the coercivity of the bilinear form a but only assume that a satisfies the conditions of Babushka Theorem. In particular, we assume that there exists $\alpha > 0$ such that

$$\inf_{u \in H_0^1(\Omega)} \sup_{v \in H_0^1(\Omega)} \frac{a(u,v)}{\|u\|_{1,\Omega}\|v\|_{1,\Omega}} \geq \alpha \tag{7}$$

For the sake of simplicity, we restrict the presentation to simplicial, affine mesh families, say $\{\mathcal{T}_h\}_{h>0}$. Let V_h be a $H_0^1(\Omega)$-conformal approximation space based on $\mathcal{T}_h$ and a Lagrange finite element of degree k. This yields the approximate problem:

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a\left(u_h, v_h\right) = \int_\Omega f v_h, \quad \forall v_h \in V_h \end{cases} \tag{8}$$

Assuming that the exact solution is smooth enough, the following a priori error estimate holds:

$$\|u - u_h\|_{1,\Omega} \leq c \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega} \leq c' \left(\sum_{K \in \mathcal{T}_h} h_K^{2k} \|u\|_{k+1,K}^2 \right)^{\frac{1}{2}}$$

To derive an a posteriori error estimate, we use the stability property (7) to obtain

$$\alpha \|u - u_h\|_{1,\Omega} \leq \sup_{v \in H_0^1(\Omega)} \frac{a(u - u_h, v)}{\|v\|_{1,\Omega}} \leq \sup_{v \in H_0^1(\Omega)} \frac{\langle \Delta (u - u_h), v \rangle_{H^{-1}, H_0^1}}{\|v\|_{1,\Omega}}$$
$$\leq \|f + \Delta u_h\|_{-1,\Omega}$$

This yields our first a posteriori error estimate.

Proposition

Let u solve (6) and u_h solve (8). Then,

$$\|u - u_h\|_{1,\Omega} \leq \frac{1}{\alpha} \|f + \Delta u_h\|_{-1,\Omega} \tag{9}$$

The main difficulty with the a posteriori estimate (9) is that the norm $\| \cdot \|_{-1,\Omega}$ cannot be localized.

To derive a local error indicator, we still use the idea of integration by parts to eliminate the exact solution, but we perform it elementwise. Let $\mathcal{F}_h^{\mathrm{i}}$ be the set of interior faces. For $F \in \mathcal{F}_h^{\mathrm{i}}$ with $F = K_1 \cap K_2$, denote by n_1 and n_2 the outward normal to K_1 and K_2, respectively. Let $[\![\partial_n u_h]\!]$ be the jump of the normal derivative of u_h across F, i.e.,

$$[\![\partial_n u_h]\!] = \nabla u_{h|K_1} \cdot n_1 + \nabla u_{h|K_2} \cdot n_2.$$

The main result is stated in the following:

Theorem

Let u solve (6) and u_h solve (8). Assume that the family $\{\mathcal{I}_h\}_{h>0}$ is shape-regular. Then, there is c such that

$$\forall h, \quad \|u - u_h\|_{1,\Omega} \leq c \left(\sum_{K \in \mathcal{T}_h} \eta_K \left(u_h, f\right)^2 \right)^{\frac{1}{2}} \tag{10}$$

with local error indicators

$$\eta_K \left(u_h, f\right) = h_K \|f + \Delta u_h\|_{0,K} + \frac{1}{2} \sum_{F \in \mathcal{F}_K} h_F^{\frac{1}{2}} \|[\![\partial_n u_h]\!]\|_{0,F} \tag{11}$$

where $\mathcal{F}_K$ is the set of faces of K that are not on $\partial\Omega$ and $h_F = \mathrm{diam}(F)$.

Proof

Since $a(u - u_h, v_h) = 0$ for all $v_h \in V_h$, the stability inequality (7) gives

$$\forall v_h \in V_h \quad \|u - u_h\|_{1,\Omega} \leq \frac{1}{\alpha} \sup_{v \in H_0^1(\Omega)} \frac{a(u - u_h, v - v_h)}{\|v\|_{1,\Omega}}.$$

We can expand the numerator of the right-hand side as follows:

$$\begin{aligned} a(u - u_h, v - v_h) &= \int_\Omega (-\Delta u)(v - v_h) - \nabla u_h \cdot \nabla(v - v_h) \\ &= \sum_{T \in \mathcal{T}_h} \left(\int_T (f + \Delta u_h)(v - v_h) - \sum_{e \in \partial T} \int_e (\partial_n u_h)(v - v_h) \right). \end{aligned}$$

Here, e denotes a face of an element T.

Proof

Since $v - v_h$ is zero on the boundary of Ω, the sum over e involves only the faces that are shared between two elements. These faces are thus interfaces. Since $v - v_h$ is continuous across each interface e, we have:

$$a(u-u_h, v-v_h) \leq \sum_{T \in \mathcal{T}_h} \left(\|f + \Delta u_h\|_{0,T} \|v - v_h\|_{0,T} + \sum_{e \in \mathcal{E}_T^i} \frac{1}{2} \|[\![\partial_n u_h]\!]\|_{0,e} \|v - v_h\|_{0,e} \right)$$

Here, $\mathcal{E}_T^i$ denotes the set of faces of T that are not on the boundary. Now, we choose $v_h = C_h v$, where C_h is the Clément operator. Applying the estimates from the Clément interpolant, we obtain:

$$a(u - u_h, v - v_h) \leq \sum_{T \in \mathcal{T}_h} \left(h_T \|f + \Delta u_h\|_{0,T} \|v\|_{1,V(T)} + \sum_{e \in \mathcal{E}_T^i} \frac{1}{2} |e|^{1/2} \|[\![\partial_n u_h]\!]\|_{0,e} \|v\|_{1}\right.$$

$$\leq \|v\|_{1,\Omega} \left(\sum_{T \in \mathcal{T}_h} \left(h_T^2 \|f + \Delta u_h\|_{0,T}^2 + \sum_{e \in \mathcal{E}_T^i} \frac{1}{2} |e| \|[\![\partial_n u_h]\!]\|_{0,e}^2 \right) \right)^{\frac{1}{2}}.$$

Proof

Thus, we have

$$\|u - u_h\|_{1,\Omega} \leq c \left(\sum_{T \in \mathcal{T}_h} \eta_T(u_h, f, h)^2 \right)^{\frac{1}{2}},$$

where

$$\eta_T(u_h, f, h) = h_T \|f + \Delta u_h\|_{0,T} + \frac{1}{2} \sum_{e \in \mathcal{E}_T^i} |e|^{\frac{1}{2}} \|[\![\partial_n u_h]\!]\|_{0,e}.$$

Optimality

Lemma (Bubble function)

Let $b_T \in H^1_0(T)$ a function s.t :

1. $0 \leq b_T \leq 1$
2. $\exists D \subset T$ s.t mes$D > 0$ et $b_T|_D \geq 1/2$

Let $m \in \mathbb{N}$. There exists $c_1 > 0$ and $c_2 > 0$ such that for all $\phi \in \mathbb{P}_m(T)$ we have

$$\|b_T\phi\|_{0,T} \leq \|\phi\|_{0,T} \leq c_1\|b_T^{1/2}\phi\|_{0,T} \tag{12}$$

$$|b_T\phi|_{1,T} \leq c_2 h_T^{-1}\|\phi\|_{0,T} \tag{13}$$

Prolongation Operator

Let $b_e \in H_0^1(e)$ a function s.t:

1. $0 \leq b_e \leq 1$
2. $\exists D \subset V(e)$ s.t $\text{mes} D > 0$ et $b_e|_D \geq 1/2$

Let $m \in \mathbb{N}$. There exists $c_1 > 0$ and $c_2 > 0$ such that for all function $\phi \in \mathbb{P}_m(e)$ we have

$$\|b_e\phi\|_{0,e} \leq \|\phi\|_{0,e} \leq c_1\|b_e^{1/2}\phi\|_{0,e} \tag{14}$$

$$c_2|e|^{1/2}\|\phi\|_{0,e} \leq \|b_e P_e(\phi)\|_{0,V(e)} \leq c_3|e|^{1/2}\|\phi\|_{0,e} \tag{15}$$

$$|b_e\phi|_{1,V(e)} \leq c_4|e|^{-1/2}\|\phi\|_{0,e} \tag{16}$$

$$\forall \phi \in \mathbb{P}_k(e), \quad P_e(\phi) = \begin{cases} P_{e,T}(\phi) & \text{on } T, \\ P_{e,T'}(\phi) & \text{on } T', \end{cases} \tag{17}$$

For an a posteriori estimator to be locally *efficient* or *optimal*, it is necessary to demonstrate that the indicator satisfies an inequality of the form:

$$\eta_T(u_h, f, h) \leq c \left(|u - u_h|_{1,V(T)} + h_T \inf_{v_h \in Z_{\ell h}} \|f - v_h\|_{0,V(T)} \right), \qquad (18)$$

where $Z_{\ell h}$ is a data approximation space defined by:

$$Z_{\ell h} = \{v_h \in L^2(\Omega) \mid \forall T \in \mathcal{T}_h, \quad v_{h|T} \in \mathbb{P}_\ell(T)\}. \qquad (19)$$

Theorem

Assume that the family $\mathcal{T}_h$ is regular. Then, there exists $c > 0$ such that

$$\eta_T(u_h, f) \leq c \left(\|u - u_h\|_{1,\Omega} + h_T \inf_{v_h \in Z_{\ell h}} \|f - v_h\|_{0,\Omega} \right).$$

Proof

In the last inequality, we used the fact that $b_T(v_h + \Delta u_h)$ is zero on the boundary of T, which allowed us to use it as a test function. Then, the inverse inequality (12) gives:

$$\begin{aligned}\|v_h + \Delta u_h\|_{0,T}^2 &\leq |u - u_h|_{1,T}\, |b_T(v_h + \Delta u_h)|_{1,T} + \|v_h - f\|_{0,T}\, \|v_h + \Delta u_h\|_{0,T} \\ &\leq \left(c h_T^{-1} |u - u_h|_{1,\Omega} + \|v_h - f\|_{0,T}\right) \|v_h + \Delta u_h\|_{0,T}.\end{aligned}$$

This gives:

$$\|v_h + \Delta u_h\|_{0,T} \leq c h_T^{-1} |u - u_h|_{1,T} + \|v_h - f\|_{0,T}. \tag{20}$$

Thus, since v_h is arbitrary, we obtain:

$$h_T \|f + \Delta u_h\|_{0,T} \leq C \left(|u - u_h|_{1,T} + h_T \inf_{v_h \in Z_{h\ell}} \|v_h - f\|_{0,T} \right). \tag{21}$$

To bound the second term

$$\frac{1}{2} \sum_{e \in \mathcal{E}_T^i} |e|^{\frac{1}{2}} \| [\![\partial_n u_h]\!] \|_{0,e},$$

since the term $[\![\partial_n u_h]\!]_{|e}$ belongs to $\mathbb{P}_k(e)$, the prolongation operator gives:

$$\begin{aligned} c \| [\![\partial_n u_h]\!] \|_{0,e}^2 &\leq \| b_e^{1/2} [\![\partial_n u_h]\!] \|_{0,e}^2 = \int_e [\![\partial_n u_h]\!] \left(b_e \, [\![\partial_n u_h]\!] \right) \\ &\leq \int_e [\![\partial_n (u_h - u)]\!] \left(b_e \, P_e([\![\partial_n u_h]\!]) \right) \qquad (\text{since } [\![\partial_n u]\!]_{|e} = 0). \end{aligned}$$

Next, we use the fact that $b_e \, P_e([\![\partial_n u_h]\!])$ is a test function for u, which is zero on the boundary of $V(e)$.

We obtain

$$
\begin{aligned}
c\|\llbracket \partial_n u_h \rrbracket\|_{0,e}^2 &\leq \int_{V(e)} \nabla(u - u_h)\nabla(b_e P_e(\llbracket \partial_n u_h \rrbracket)) + \sum_{T \in V(e)} \int_T b_e P_e(\llbracket \partial_n u_h \rrbracket)\Delta \\
&\leq |u - u_h|_{V(e)}\, e^{-1/2}\, \|\llbracket \partial_n u_h \rrbracket\|_{0,e} + \sum_{T \in V(e)} \|b_e P_e(\llbracket \partial_n u_h \rrbracket)\|_{0,T}\, \|f
\end{aligned}
$$

We again use the previously obtained bound for $\|f + \Delta u_h\|_{0,T}$, and applying the bubble function estimate, we obtain:

$$
e^{1/2}\|\llbracket \partial_n u_h \rrbracket\|_{0,e} \leq C \left(|u - u_h|_{1,V(e)} + h_T \inf_{v_h \in Z_{h\ell}} \|v_h - f\|_{0,V(e)} \right). \tag{22}
$$