Lecture 3: A posteriori error analysis by flux reconstruction

Pr. Ismail Merabet

Univ. of K-M-Ouargla

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Contents

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$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \quad\;\; u = 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

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$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ such that} \\ (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \end{cases} \tag{2}$$

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The existence and uniqueness of a solution of (2) is ensured by the Riesz representation theorem (or by the Lax-Milgram theorem).

Flux

Definition. (Flux)

Let u be the solution of (2). Set

$$\sigma := -\nabla u \tag{3}$$

We will call $\boldsymbol{\sigma}$ the flux.

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We recall the definitions of the spaces $H_0^1(\Omega)$ and $\mathbf{H}(\text{div}, \Omega)$, we have:

$$H_0^1(\Omega) := \{v \in L^2(\Omega); \nabla v \in (L^2(\Omega))^d; \gamma_0(v) := v|_{\partial\Omega} = 0\}$$

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Theorem 1. (Properties of the weak solution)

Let u be the solution of (2). Let $\boldsymbol{\sigma}$ be given by (3).Then

$$u \in H_0^1(\Omega), \quad \boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div}, \Omega), \quad \mathrm{div}\ \boldsymbol{\sigma} = f$$

Prager-Synge equality

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$$\|\nabla (u - u_h)\|^2 + \|\nabla u + \boldsymbol{\sigma}_h\|^2 = \|\nabla u_h + \boldsymbol{\sigma}_h\|^2 \tag{4}$$

Proof.

Adding and subtracting ∇u, we develop

$$\begin{aligned}
\|\nabla u_h + \boldsymbol{\sigma}_h\|^2 &= \|\nabla (u_h - u) + \nabla u + \boldsymbol{\sigma}_h\|^2 \\
&= \|\nabla (u_h - u)\|^2 + \|\nabla u + \boldsymbol{\sigma}_h\|^2 + 2\left(\nabla (u_h - u), \nabla u + \boldsymbol{\sigma}_h\right)
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But

$\left(\nabla(u_h - u), \nabla u + \boldsymbol{\sigma}_h\right) = \left(u_h - u, -\,\mathrm{div}\ (\nabla u + \boldsymbol{\sigma}_h)\right) = (u_h - u, f - f) = 0$

whence the assertion follows.

Remark

Under the assumptions of previous Theorem, it follows from (4) that

$$\|\nabla (u - u_h)\| \leq \|\nabla u_h + \boldsymbol{\sigma}_h\| \tag{5}$$

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$$\text{div } \boldsymbol{\sigma}_h = \Pi_{Q_h} f.$$

Here $Q_h \subset L^2(\Omega)$ and $\Pi_{Q_h} f$ is the $L^2(\Omega)$-orthogonal projection onto Q_h.

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$$\mathrm{div}\ \boldsymbol{\sigma}_h = \Pi_{Q_h} f.$$

Here $Q_h \subset L^2(\Omega)$ and $\Pi_{Q_h} f$ is the $L^2(\Omega)$-orthogonal projection onto Q_h. Then the remaining difference between f and $\Pi_{Q_h} f$ can be treated, giving rise to the so-called data oscillation.

Next, we will be inspired by the above result.

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Importantly, the construction of $\boldsymbol{\sigma}_h$ will be local, over patches of mesh elements, in contrast to some initial developments where a costly global solve over the entire domain Ω was necessary.

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Importantly, the construction of $\boldsymbol{\sigma}_h$ will be local, over patches of mesh elements, in contrast to some initial developments where a costly global solve over the entire domain Ω was necessary. We will also directly treat nonconforming approximate solutions not satisfying the assumptions of Theorem 2 but merely verifying $u_h \in H^1(\mathcal{T}_h)$.

Approximate solution

In order to make the presentation general, not restricted to any particular numerical method, we are led to suppose here that the approximate solution u_h that we are given satisfies

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$$u_h \in H^1(\mathcal{T}_h) \tag{6}$$

where $H^1(\mathcal{T}_h)$ is the broken Sobolev space.

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where $H^1(\mathcal{T}_h)$ is the broken Sobolev space.
In analogy with the 1d case we set:

Definition (Approximate flux)

Let u_h be the approximate solution of u. We will call

$$-\nabla u_h \tag{7}$$

the approximate flux.

Potential reconstruction

Remark. (Properties of the approximate solution u_h)

Let u_h be the approximate solution (see 6). Then

$$u_h \notin H_0^1(\Omega), \quad -\nabla u_h \notin \boldsymbol{H}(\mathrm{div}, \Omega), \quad \mathrm{div}\ (-\nabla u_h) \neq f \quad \text{in general.} \tag{8}$$

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Definition. (Potential reconstruction)

Let u_h be the approximate solution (see (6)). We will call the potential reconstruction any function s_h constructed from u_h which satisfies

$$s_h \in H_0^1(\Omega) \tag{9}$$

In order to obtain satisfactory result we will impose that the flux reconstruction $\boldsymbol{\sigma}_h$ lies in the correct functional space, but we will also prescribe a condition on its divergence.

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In order to obtain satisfactory result we will impose that the flux reconstruction $\boldsymbol{\sigma}_h$ lies in the correct functional space, but we will also prescribe a condition on its divergence. This is linked to the fact that on the continuous level, div $\boldsymbol{\sigma} = f$.

Equilibrated flux reconstruction

Definition. (Equilibrated flux reconstruction)

We will call the equilibrated flux reconstruction any function $\boldsymbol{\sigma}_h$ constructed from u_h which satisfies

$$\boldsymbol{\sigma}_h \in \mathbf{H}(\mathrm{div}, \Omega), \tag{10}$$

$$(\mathrm{div}\ \boldsymbol{\sigma}_h, 1)_K = (f, 1)_K \quad \forall K \in \mathcal{T}_h \tag{11}$$

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We give here a posteriori error estimate on the distance between u, the unknown solution of (2), and u_h, the known approximate solution characterized by (6). Note that it gives a guaranteed upper bound.

Theorem. (A general a posteriori error estimate for (1)-(**??**))

Let u be the weak solution of (2). Let u_h be an arbitrary function satisfying (6). Let s_h be a potential reconstruction and $\boldsymbol{\sigma}_h$ an equilibrated flux reconstruction For any $K \in \mathcal{T}_h$, define the indicators:

$$\eta_{\mathrm{R},K} := \frac{h_K}{\pi} \|f - \operatorname{div} \boldsymbol{\sigma}_h\|_K , \quad \eta_{\mathrm{F},K} := \|\nabla u_h + \sigma_h\|_K , \tag{12}$$

and the nonconformity estimator by

$$\eta_{\mathrm{NC},K} := \|\nabla (u_h - s_h)\|_K \tag{13}$$

Then

$$\|\nabla (u - u_h)\|^2 \leq \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{F},K} + \eta_{\mathrm{R},K})^2 + \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \tag{14}$$

Proof.

Let us define a function $s \in H_0^1(\Omega)$ by

$$(\nabla s, \nabla v) = (\nabla u_h, \nabla v) \quad \forall v \in H_0^1(\Omega) \tag{15}$$

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Let us define a function $s \in H_0^1(\Omega)$ by

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Then,

$$\|\nabla (u - u_h)\|^2 = \|\nabla(u - s)\|^2 + \|\nabla (s - u_h)\|^2 \tag{16}$$

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This follows from the fact that:

$$\begin{aligned} \|\nabla (u - u_h)\|^2 &= \|\nabla (u - s + s - u_h)\|^2 \\ &= \|\nabla(u - s)\|^2 + \|\nabla (s - u_h)\|^2 + \underbrace{2 (\nabla(u - s), \nabla (s - u_h))}_{=0} \end{aligned}$$

Proof cont...

Moreover,

$$\left\|\nabla\left(s-u_h\right)\right\|^2=\min_{w\in H_0^1(\Omega)}\left\|\nabla\left(w-u_h\right)\right\|^2 \tag{17}$$

Proof cont...

Moreover,

$$\left\| \nabla \left(s - u_h \right) \right\|^2 = \min_{w \in H_0^1(\Omega)} \left\| \nabla \left(w - u_h \right) \right\|^2 \tag{17}$$

Indeed, we actually have the property (16) for any function $w \in H_0^1(\Omega)$ (any other information about u than $u \in H_0^1(\Omega)$ was not used),

$$\left\| \nabla \left(w - u_h \right) \right\|^2 = \left\| \nabla (w - s) \right\|^2 + \left\| \nabla \left(s - u_h \right) \right\|^2 \tag{18}$$

from where do we get

$$\left\| \nabla \left(s - u_h \right) \right\|^2 = \left\| \nabla \left(w - u_h \right) \right\|^2 - \left\| \nabla (w - s) \right\|^2 \leq \left\| \nabla \left(w - u_h \right) \right\|^2 \tag{19}$$

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Moreover,

$$\|\nabla (s - u_h)\|^2 = \min_{w \in H_0^1(\Omega)} \|\nabla (w - u_h)\|^2 \tag{17}$$

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Returning back to a posteriori analysis, it follows from (17) that, for the potential reconstruction s_h we have the bound

$$\|\nabla (s - u_h)\|^2 \leq \|\nabla (s_h - u_h)\|^2 = \sum_{K \in \mathcal{T}_h} \eta_{\mathrm{NC},K}^2 \tag{20}$$

Proof cont...

On the other hand, we have

$$
\begin{aligned}
\|\nabla(u-s)\| &= \sup_{\varphi\in H_0^1(\Omega);\|\nabla\varphi\|=1} (\nabla(u-s),\nabla\varphi) \\
&= \sup_{\varphi\in H_0^1(\Omega);\|\nabla\varphi\|=1} \left(\nabla\left(u-u_h\right),\nabla\varphi\right)
\end{aligned}
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Let $\varphi \in H_0^1(\Omega)$ with $\|\nabla\varphi\| = 1$ be fixed. Using (2) of the weak solution, we have

$$\left(\nabla\left(u-u_h\right),\nabla\varphi\right) = (f,\varphi) - (\nabla u_h,\nabla\varphi) \tag{21}$$

and adding and subtracting $(\boldsymbol{\sigma}_h,\nabla\varphi)$, where $\boldsymbol{\sigma}_h$ is the equilibrated flux reconstruction and using the Green theorem

$$(\sigma_h,\nabla\varphi) = -\left(\operatorname{div}\,\sigma_h,\varphi\right),$$

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we have

$$\left(\nabla\left(u-u_h\right),\nabla\varphi\right) = (f - \text{div } \boldsymbol{\sigma}_h, \varphi) - (\nabla u_h + \boldsymbol{\sigma}_h, \nabla\varphi) \tag{22}$$

Proof cont...

The Cauchy-Schwarz inequality gives

$$
\begin{aligned}
-(\nabla u_h + \sigma_h, \nabla\varphi) &\leq \sum_{K\in\mathcal{T}_h} \|\nabla u_h + \sigma_h\|_K \|\nabla\varphi\|_K \\
&= \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{F},K} \|\nabla\varphi\|_K
\end{aligned}
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Proof cont...

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whereas the approximate equilibrium property (11), the Poincaré inequality, and the Cauchy-Schwarz inequality give

$$(f - \operatorname{div} \boldsymbol{\sigma}_h, \varphi) = \sum_{K\in\mathcal{T}_h} (f - \operatorname{div} \boldsymbol{\sigma}_h, \varphi)_K = \sum_{K\in\mathcal{T}_h} (f - \operatorname{div} \boldsymbol{\sigma}_h, \varphi - \varphi_K)_K$$

$$\leq \sum_{K\in\mathcal{T}_h} \frac{h_K}{\pi} \|f - \operatorname{div} \boldsymbol{\sigma}_h\|_K \|\nabla\varphi\|_K = \sum_{K\in\mathcal{T}_h} \eta_{\mathrm{R},K}\|\nabla\varphi\|_K$$

Proof cont...

Combining the above results while using the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
\|\nabla(u-s)\|^2 &\le \left(\sup_{\varphi \in H_0^1(\Omega); \|\nabla\varphi\|=1} \left\{ \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{F},K} + \eta_{\mathrm{R},K}) \|\nabla\varphi\|_K \right\} \right)^2 \\
&\le \sum_{K \in \mathcal{T}_h} (\eta_{\mathrm{F},K} + \eta_{\mathrm{R},K})^2
\end{aligned}
$$

whence the assertion of the theorem follows.