Lecture 1: Variational formulation and Sobolev spaces

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Contents

Model problem in 1d

Consider the following boundary value problem

$$\begin{cases} -u'' = f \text{ in }]0,1[\\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

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then integration by parts yields

$$(f, v) = \int_0^1 u'(x)v'(x)\ dx =: a(u, v). \tag{3}$$

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Problem (4) is called *the weak formulation* or the variational problem of problem (1).

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By construction it is clear that any solution of (1) is a solution of (4). Of course, the central issue is that (4) gives a solution of (1).

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Suppose that $f \in C^0([0,1])$ and $u \in C^2([0,1])$ satisfies (4). Then u is a solution of (1).

Proof

Let $v \in V \cap C^1([0,1])$, then

$$(f, v) = \int_\Omega u'v' \, dx = -\int_\Omega u''v \, dx + u'v|_0^1 = (-u'', v).$$

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So,

$$(f + u'', v) = 0, \quad \forall v \in V \cap C^1([0,1]). \tag{5}$$

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take

$$v(x) = (x - x_0)^2 (x - x_1)^2$$

this implies that $(f + u'', v) \neq 0$ which is in contradiction with (5).

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- Observe that if the space if with finite dimension then the weak formulation (4) can be reformulated as a linear system $Ax = b$. in that case, the uniqueness means that A is injective thus it is an isomorphisme this implies the existence also.
- But we work here in infinite dimensional case, and the previous theory does not work here. The existence of solution follows from the Lax-Milgram theorem which suppose that the space V must be a Hilbert space.

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(Green's Formula I)

Let $v \in C^1(\bar{\Omega})$ with compact support in $\bar{\Omega}$. Then we have

$$\int_\Omega \frac{\partial v}{\partial x_i}(x)\, dx = \int_{\partial\Omega} v(x) n_i(x)\, ds \tag{6}$$

where n_i is the i-th component of the unit outward normal to Ω.

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$$\int_\Omega u \frac{\partial v}{\partial x_i}(x)\ dx = -\int_\Omega \frac{\partial u}{\partial x_i}(x)\ v(x)\ dx + \int_{\partial\Omega} u(x) v(x) n_i(x)\ ds \tag{7}$$

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$$-\int_\Omega \Delta u \; v \; dx = \int_\Omega \nabla u \cdot \nabla v \; dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} \; v \; ds \tag{8}$$

Proof

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Given a function $v \in L^2(\Omega)$. We say that v has a weak derivative if there exists $w_i \in L^2(\Omega)$, for all $i = 1, ..., N$ such that for all $\phi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega v \frac{\partial \phi}{\partial x_i} = -\int_\Omega w_i \phi \, dx$$

the functions w_i are called the weak derivatives and they are denoted by $\frac{\partial v}{\partial x_i}$.

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Let Ω be an open set of $\mathbb{R}^N$. The Sobolev space $H^1(\Omega)$ is given by :

$$H^1(\Omega) =: \{u \in L^2(\Omega); \frac{\partial u}{\partial x_i} \in L^2(\Omega), i = 1, 2, ..., N\} \tag{9}$$

where $\dfrac{\partial u}{\partial x_i}$ is the weak derivative of u

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Definition

for $m \in \mathbb{N}$

$$H^m(\Omega) =: \{u \in \mathcal{D}'(\Omega); D^\alpha u \in L^2(\Omega) \quad |\alpha| \le m\} \tag{10}$$

For $m = 0$ we have $H^0(\Omega) = L^2(\Omega)$ and for $m = 1$, $H^1(\Omega)$.

Sobolev space $H^m(\Omega)$

We equiped $H^m(\Omega)$ by the inner product:

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Proposition

1. If $m \geq m'$, $H^m(\Omega)$ is continuously embedded in $H^{m'}(\Omega)$.
2. $H^m(\Omega)$ equiped with the inner product (11) is a Hilbert spacce.

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$$\|D^m u\|_{L^\infty(\Omega)} \leq C\|u\|_{k,\Omega} \tag{13}$$

In addition there exists a function of class C^m equal to u almost every where.

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$$\|u\|_{L^2(\partial\Omega)} \leq c \ \|u\|_{H^1(\Omega)}, \forall u \in H^1(\Omega). \tag{15}$$

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Theorem

The $H_0^1(\Omega)$ is the kernel of γ_0, i.e.,

$$H_0^1(\Omega) = \{u \in H^1(\Omega), u_{/\partial\Omega} = 0\}$$

Poincaré's inequality

Lemma

Let Ω a bounded set of $\mathbb{R}^N$. Then, there exists a positive C which depends only on Ω such that:

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C\|\nabla v\|_{L^2(\Omega)}. \tag{16}$$