MEF I
Lecture 2: Well-posed problems

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In this course, we will see three main theorems regarding the well-posedness of the linear variational problem: for $F \in V^*$,

$$\text{find } u \in V \text{ such that } a(u, v) = F(v) \quad \text{for all } v \in V$$

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In this lecture we will study the first two.

Definition (Bounded bilinear form)

A bilinear form $a : H \times H \to \mathbb{R}$ is said be to *bounded* if there exists $C \in [0, \infty)$ such that

$$|a(v, w)| \leq C \|v\|_H \|w\|_H \quad \text{for all } v, w \in H.$$

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As with linear functionals, this is equivalent to continuity.

The best constant C satisfying the definition is called the continuity constant of a:

$$C := \sup_{\substack{v \in H \\ v \neq 0}} \sup_{\substack{w \in H \\ w \neq 0}} \frac{|a(v, w)|}{\|v\|_H\|w\|_H}.$$

Definition (Coercive bilinear form)

A bilinear form $a : H \times H \to \mathbb{R}$ is said be to *coercive* on $V \subset H$ or V-coercive if there exists $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|_H^2 \quad \text{for all } v \in V.$$

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$$a(v,v) \geq \alpha \|v\|_H^2 \quad \text{for all } v \in V.$$

As before, the best constant α satisfying the definition is called the coercivity constant of a:

$$\alpha := \inf_{\substack{u \in V \\ u \neq 0}} \frac{a(u,u)}{\|u\|_H^2}.$$

Note that we must have $\alpha \leq C$, as

$$\alpha \|u\|_H^2 \leq a(u, u) \leq C \|u\|_H^2.$$

Let's assume for now that a is also symmetric.

Theorem

Let H be a Hilbert space, and suppose $a : H \times H \to \mathbb{R}$ is a symmetric bilinear form that is continuous on H and coercive on a closed subspace $V \subset H$. Then $(V, a(\cdot,\cdot))$ is a Hilbert space.

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Let H be a Hilbert space, and suppose $a : H \times H \to \mathbb{R}$ is a symmetric bilinear form that is continuous on H and coercive on a closed subspace $V \subset H$. Then $(V, a(\cdot, \cdot))$ is a Hilbert space.

We must prove that a is an inner product on V, and that V is complete with respect to the induced norm.

If $0 = a(v,v) \geq \alpha\|v\|_H^2 \geq 0$, then $v = 0$. Clearly $a(v,v) \geq 0$ for all $v \in V$. Symmetry and linearity are assumed, so $a(\cdot,\cdot)$ is an inner product on V.

Denote

$$\|v\|_a = \sqrt{a(v,v)}.$$

It remains to show that $(V, \|\cdot\|_a)$ is complete.

Suppose that $\{v_n\}$ is a Cauchy sequence in $(V, \|\cdot\|_a)$, i.e.

$$\forall \varepsilon > 0 \, \exists \, N > 0 \, \forall \, m, n > N \, \|v_n - v_m\|_a < \varepsilon.$$

Since $\|v\|_H \leq \frac{1}{\sqrt{\alpha}} \|v\|_a$, $\|v_n - v_m\|_H < \varepsilon / \sqrt{\alpha}$ and $\{v_n\}$ is also Cauchy in $(H, \|\cdot\|_H)$.

Since H is complete, there exists $v \in H$ such that $v_n \to v$ in the $\|\cdot\|_H$ norm. Since V is closed in H, $v \in V$. Now observe that as a is bounded

$$\|v - v_n\|_a = \sqrt{a(v - v_n, v - v_n)} \leq \sqrt{C \|v - v_n\|_H^2} = \sqrt{C} \|v - v_n\|_H$$

where C is the continuity constant for a. Hence $v_n \to v$ in the $\|\cdot\|_a$ norm too, so V is complete with respect to this norm.

Faster: note that coercivity and continuity guarantee that

$$\alpha\|v\|_H^2 \leq \|v\|_a^2 \leq C\|v\|_H^2 \quad \text{for all } v \in V.$$

So the norms are equivalent, and hence induce the same notion of convergence and completeness.

The well-posedness of the symmetric coercive bounded linear variational problem follows immediately.

Theorem

Let V be a closed subspace of a Hilbert space H. Let $a : H \times H \to \mathbb{R}$ be a symmetric continuous V-coercive bilinear form, and let $F \in V^*$. Consider the variational problem:

$$\text{find } u \in V \text{ such that } a(u, v) = F(v) \quad \text{for all } v \in V.$$

This problem has a unique stable solution.

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This problem has a unique stable solution.

Proof.

Our previous result implies that $a(\cdot, \cdot)$ is an inner product on V, and that (V, a) is a Hilbert space. Apply the Riesz Representation Theorem, that every bounded linear functional $F \in V^*$ has a unique representative (in this case u).

Proof.

Stability means that we can find a constant c such that

$$\|u\|_V \leq c\|F\|_{V^*}.$$

By the Riesz representation theorem, the Riesz map is an isomorphism, so this follows for the norms generated by the inner product with $c = 1$. □

Example

The variational problem

$$\text{find } u \in H_0^1(\Omega) \text{ such that } \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x \text{ for all } v \in H_0^1(\Omega)$$

is well-posed, as $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$, and we will show later that the bilinear form is $H_0^1(\Omega)$-coercive, symmetric, and bounded.

Section 3

The nonsymmetric case

Now let us drop the assumption that $a(u, v) = a(v, u)$.

Theorem (Lax–Milgram)

Let V be a closed subspace of a Hilbert space H. Let $a : H \times H \to \mathbb{R}$ be a (not necessarily symmetric) continuous V-coercive bilinear form, and let $F \in V^*$. Consider the variational problem:

$$\text{find } u \in V \text{ such that } a(u, v) = F(v) \quad \text{for all } v \in V.$$

This problem has a unique stable solution.

For the proof, it will be more convenient to treat the LVP as an equation in the dual V^*.

Lemma

Let $a : V \times V \to \mathbb{R}$ be linear in its second argument and bounded. For any $u \in V$, define a functional via $A : u \mapsto Au$

$$(Au)(v) := a(u, v) \quad \text{for all } v \in V.$$

Then $Au \in V^*$, i.e. $A : V \to V^*$. Furthermore, A is itself linear if a is linear in its first argument.

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Proof.

Linearity is straightforward. For boundedness (so that $Au \in V^*$),

$$\|Au\|_{V^*} = \sup_{v \neq 0} \frac{|Au(v)|}{\|v\|_H} = \sup_{v \neq 0} \frac{|a(u, v)|}{\|v\|_H} \leq C\|u\|_H < \infty.$$

$\square$

Thus, the variational problem

$$\text{find } u \in V \text{ such that } a(u, v) = F(v) \quad \text{for all } v \in V$$

is equivalent to

$$\text{find } u \in V \text{ such that } \langle Au, v \rangle = \langle F, v \rangle \quad \text{for all } v \in V.$$

And since equality of two dual objects means exactly that they have the same output on all possible inputs, this is equivalent to

$$\text{find } u \in V \text{ such that } Au = F,$$

where the equality is between dual objects, $Au \in V^*$ and $F \in V^*$.

Example

In the case of the homogeneous Dirichlet Laplacian operator, we have $A : H_0^1(\Omega) \to \left(H_0^1(\Omega)\right)^*$. We could symbolically write $A = -\nabla^2$ and interpret

$$-\nabla^2 u = f$$

as an equation in the dual of $H_0^1(\Omega)$. This dual space is denoted

$$H^{-1}(\Omega) := \left(H_0^1(\Omega)\right)^*$$

and we can regard the Laplacian as a map $H_0^1(\Omega) \to H^{-1}(\Omega)$.

We know from the Riesz Representation Theorem that there is an isometric isomorphism $\mathcal{R} : V^* \to V$ from the dual of a Hilbert space V^* back to V. By composing these operators, we have the problem

$$\text{find } u \in V \text{ such that } \mathcal{R}Au = \mathcal{R}F,$$

where the equality is between *primal* objects, $\mathcal{R}Au \in V$ and $\mathcal{R}F \in V$.

Proof strategy: we will define a map $T : V \to V$ whose fixed point is the solution of our variational problem, and then show it is a contraction, and invoke the Banach contraction mapping theorem.

Theorem (Contraction mapping theorem)

Given a nonempty Banach space V and a mapping $T : V \to V$ satisfying

$$\|Tv_1 - Tv_2\| \leq M\|v_1 - v_2\|$$

for all $v_1, v_2 \in V$ and fixed M, $0 \leq M < 1$, there exists a unique $u \in V$ such that

$$u = Tu.$$

That is, a contraction T has a unique fixed point u.

We now prove the Lax–Milgram Theorem.

Proof.

Cast the variational problem

$$\text{find } u \in V \text{ such that } a(u,v) = F(v) \quad \text{for all } v \in V$$

as the primal equality

$$\text{find } u \in V \text{ such that } \mathcal{R}Au = \mathcal{R}F$$

as discussed. For a fixed $\rho \in (0, \infty)$, define the affine map $T : V \to V$

$$Tv = v - \rho \left(\mathcal{R}Av - \mathcal{R}F \right).$$

If T is a contraction for some ρ, then there exists a unique fixed point $u \in V$ such that

$$Tu = u - \rho(\mathcal{R}Au - \mathcal{R}F) = u,$$

i.e. that $\mathcal{R}Au = \mathcal{R}F$. We now show that such a ρ exists.

Proof.

For any $v_1, v_2 \in V$, let $v = v_1 - v_2$. Then

$$\|Tv_1 - Tv_2\|_H^2 = \|v_1 - v_2 - \rho(\mathcal{R}Av_1 - \mathcal{R}Av_2)\|_H^2$$

Proof.

For any $v_1, v_2 \in V$, let $v = v_1 - v_2$. Then

$$\begin{aligned} \|Tv_1 - Tv_2\|_H^2 &= \|v_1 - v_2 - \rho(\mathcal{R}Av_1 - \mathcal{R}Av_2)\|_H^2 \\ &= \|v - \rho(\mathcal{R}Av)\|_H^2 && \text{(lin. of } \mathcal{R}\text{, } A\text{)} \end{aligned}$$

Proof.

For any $v_1, v_2 \in V$, let $v = v_1 - v_2$. Then

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\begin{aligned}
\|Tv_1 - Tv_2\|_H^2 &= \|v_1 - v_2 - \rho(\mathcal{R}Av_1 - \mathcal{R}Av_2)\|_H^2 \\
&= \|v - \rho(\mathcal{R}Av)\|_H^2 && \text{(lin. of } \mathcal{R}, A) \\
&= \|v\|_H^2 - 2\rho(\mathcal{R}Av, v) + \rho^2\|\mathcal{R}Av\|_H^2 && \text{(lin. of i. prod.)}
\end{aligned}
$$

Proof.

For any $v_1, v_2 \in V$, let $v = v_1 - v_2$. Then

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&= \|v - \rho(\mathcal{R}Av)\|_H^2 && \text{(lin. of } \mathcal{R}\text{, } A\text{)} \\
&= \|v\|_H^2 - 2\rho(\mathcal{R}Av, v) + \rho^2\|\mathcal{R}Av\|_H^2 && \text{(lin. of i. prod.)} \\
&= \|v\|_H^2 - 2\rho Av(v) + \rho^2 Av(\mathcal{R}Av) && \text{(definition of } \mathcal{R}\text{)}
\end{aligned}$$

Proof.

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&= \|v\|_H^2 - 2\rho(\mathcal{R}Av, v) + \rho^2\|\mathcal{R}Av\|_H^2 && \text{(lin. of i. prod.)} \\
&= \|v\|_H^2 - 2\rho Av(v) + \rho^2 Av(\mathcal{R}Av) && \text{(definition of } \mathcal{R}\text{)} \\
&= \|v\|_H^2 - 2\rho a(v,v) + \rho^2 a(v, \mathcal{R}Av) && \text{(definition of } A\text{)}
\end{aligned}$$

Proof.

For any $v_1, v_2 \in V$, let $v = v_1 - v_2$. Then

$$\begin{aligned}
\|Tv_1 - Tv_2\|_H^2 &= \|v_1 - v_2 - \rho(\mathcal{R}Av_1 - \mathcal{R}Av_2)\|_H^2 \\
&= \|v - \rho(\mathcal{R}Av)\|_H^2 && \text{(lin. of } \mathcal{R}, A) \\
&= \|v\|_H^2 - 2\rho(\mathcal{R}Av, v) + \rho^2\|\mathcal{R}Av\|_H^2 && \text{(lin. of i. prod.)} \\
&= \|v\|_H^2 - 2\rho Av(v) + \rho^2 Av(\mathcal{R}Av) && \text{(definition of } \mathcal{R}) \\
&= \|v\|_H^2 - 2\rho a(v, v) + \rho^2 a(v, \mathcal{R}Av) && \text{(definition of } A) \\
&\leq \|v\|_H^2 - 2\rho\alpha\|v\|_H^2 + \rho^2 C\|v\|_H\|\mathcal{R}Av\|_H && \text{(coerc. \& cont.)}
\end{aligned}$$

Proof.

For any $v_1, v_2 \in V$, let $v = v_1 - v_2$. Then

$$\begin{aligned}
\|Tv_1 - Tv_2\|_H^2 &= \|v_1 - v_2 - \rho(\mathcal{R}Av_1 - \mathcal{R}Av_2)\|_H^2 \\
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&= \|v\|_H^2 - 2\rho a(v, v) + \rho^2 a(v, \mathcal{R}Av) && \text{(definition of } A\text{)} \\
&\leq \|v\|_H^2 - 2\rho\alpha\|v\|_H^2 + \rho^2 C\|v\|_H\|\mathcal{R}Av\|_H && \text{(coerc. \& cont.)} \\
&\leq (1 - 2\rho\alpha + \rho^2 C^2)\|v\|_H^2 && \text{(} A \text{ cts, } \mathcal{R} \text{ isom.)}
\end{aligned}$$

Proof.

For any $v_1, v_2 \in V$, let $v = v_1 - v_2$. Then

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&= \|v\|_H^2 - 2\rho a(v, v) + \rho^2 a(v, \mathcal{R}Av) && \text{(definition of } A) \\
&\leq \|v\|_H^2 - 2\rho\alpha\|v\|_H^2 + \rho^2 C\|v\|_H\|\mathcal{R}Av\|_H && \text{(coerc. \& cont.)} \\
&\leq (1 - 2\rho\alpha + \rho^2C^2)\|v\|_H^2 && (A \text{ cts, } \mathcal{R} \text{ isom.}) \\
&= (1 - 2\rho\alpha + \rho^2C^2)\|v_1 - v_2\|_H^2.
\end{aligned}$$

Proof.

Thus, if we can find a ρ such that

$$1 - 2\rho\alpha + \rho^2 C^2 < 1,$$

i.e. that

$$\rho(\rho C^2 - 2\alpha) < 0,$$

then we are done. If we choose $\rho \in (0, 2\alpha/C^2)$ then T is a contraction and a unique solution exists.

Proof.

It remains to show stability.

$$\|u\|_H^2 \leq \frac{1}{\alpha} a(u,u) = \frac{1}{\alpha} F(u) \leq \frac{1}{\alpha} \|F\|_{V^*} \|u\|_H,$$

and so

$$\|u\|_H \leq \frac{1}{\alpha} \|F\|_{V^*}$$

□