Lecture 2: Approximation by trigonometric Polynomials: Jackson's theorem

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Contents

Introduction

In this lecture we address the question of how quickly a given continuous function can be approximated by a sequence of polynomials of increasing degree. The results were mostly obtained by Dunham Jackson in the first third of the twentieth century and are known collectively as Jackson's theorems. Essentially they say that if a function is in C^k then it can be approximated by a sequence of polynomials of degree n in such a way that the error is at most C/n^k as $n \to \infty$. Thus the smoother a function is, the better the rate of convergence.
Jackson proved this sort of result both for approximation by polynomials and for approximation by trigonometric polynomials (finite Fourier series). The two sets of results are intimately related, as we shall see, but it is easier to get started with the results for trigonometric polynomials, as we do now.

Let $C_{2\pi}$ be the set of 2π-periodic continuous functions on the real line, and $C_{2\pi}^k$ the set of 2π-periodic functions which belong to $C^k(\mathbb{R})$. We shall investigate the rate of approximation of such functions by trigonometric polynomials of degree n. By these we mean linear combinations of the $n+1$ functions :

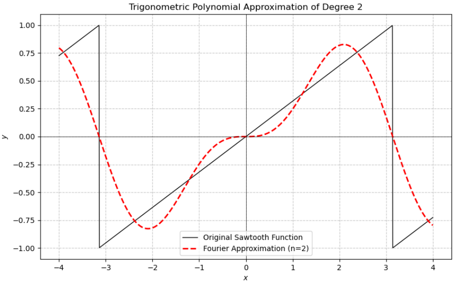
$$1, \quad \cos kx, \quad \sin kx, \qquad k = 1, \ldots, n,$$

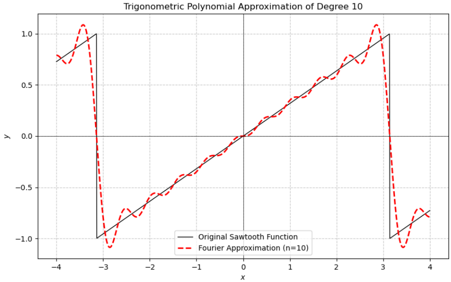
and we denote by $\mathcal{T}_n$ the space of all trigonometric polynomials of degree n, i.e., the span of these of the $2n+1$ functions. Using the relations:

$$\sin x = \left(e^{ix} - e^{-ix}\right)/(2i), \quad \cos x = \left(e^{ix} + e^{-ix}\right)/2,$$

we can equivalently write

$$\mathcal{T}_n = \left\{ \sum_{k=-n}^{n} c_k e^{ikx} \mid c_k \in \mathbb{C}, \quad c_{-k} = \bar{c}_k \right\}$$


Trigonometric Polynomial Approximation of Degree 2
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
y
−4
−3
−2
−1
0
1
2
3
4
x
Original Sawtooth Function
Fourier Approximation (n=2)


Trigonometric Polynomial Approximation of Degree 10
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0.5
0.0
−0.5
−1.0
y
−4
−3
−2
−1
0
1
2
3
4
x
Original Sawtooth Function
Fourier Approximation (n=10)

The Jackson Theorem for the approximation of functions in $C^1_{2\pi}$ by trigonometric polynomials.

Theorem

If $f \in C^1_{2\pi}$, then

$$\inf_{p \in \mathcal{T}_n} \|f - p\|_\infty \leq \frac{\pi}{2(n+1)} \left\|f'\right\|_\infty$$

The proof of Jackson' theorem uses two lemma.

Lemma (Riemann)

Define the piecewise constant function

$$s(y) = (-1)^k, \quad \frac{k\pi}{n+1} \leq y < \frac{(k+1)\pi}{n+1}, \quad k \in \mathbb{Z}.$$

Then $\int \sin(ky)\, s(y) dy = 0, \quad k = 1, \ldots, n,$

Lemma

There exists $q_n \in \mathcal{T}_n$ such that

$$\int_{-\pi}^{\pi} |y - q_n(y)|\, dy \leq \frac{\pi^2}{n+1}$$

$$q_n(\frac{k\pi}{n+1}) = \frac{k\pi}{n+1}, \quad k = -n, \ldots, n$$

Proof of Jacksons's theorem

We start by writing $f(x)$ as an integral of f' times an appropriate kernel. Consider the integral

$$\int_{-\pi}^{\pi} yf'(x+\pi+y)dy.$$

Integrating by parts and using the fact that f is 2π-periodic we get

$$\int_{-\pi}^{\pi} yf'(x+\pi+y)dy = -\int_{-\pi}^{\pi} f(x+\pi+y)dy + 2\pi f(x).$$

The integral on the right-hand side is just the integral of f over one period (and so independent of x), and we can rearrange to get

$$f(x) = \bar{f} + \frac{1}{2\pi}\int_{-\pi}^{\pi} yf'(x+\pi+y)dy$$

where $\bar{f}$ is the average value of f over any period.

Now suppose we replace the function y in the last integral with a trigonometric polynomial $q_n(y) = \sum_{k=-n}^{n} c_k e^{iky}$. This gives

$$\int_{-\pi}^{\pi} q_n(y) f'(x+\pi+y)dy = \int_{-\pi}^{\pi} q_n(y-\pi-x) f'(y)dy$$
$$= \sum_{k=-n}^{n} c_k \int_{-\pi}^{\pi} e^{ik(y-\pi)} f'(y)dy e^{-ikx}$$

which is a trigonometric polynomial of degree at most n in x. Thus

$$p_n(x) := \bar{f} + \frac{1}{2\pi} \int_{-\pi}^{\pi} q_n(y) f'(x+\pi+y)dy \in \mathcal{T}_n,$$

and $p_n(x)$ is close to $f(x)$ if $q(y)$ is close to y on $[-\pi, \pi]$.

Specifically

$$
\begin{aligned}
|f(x) - p_n(x)| &= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} [y - q_n(y)] \, f'(x + \pi + y) dy \right| \\
&\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |y - q_n(y)| \, dy \, \|f'\| \, . \qquad (1)
\end{aligned}
$$

Thus to obtain a bound on the error, we need only give a bound on the L^1 error in trigonometric polynomial approximation to the function $g(y) = y$ on $[-\pi, \pi]$.

Note that, since we are working in the realm of 2π periodic functions, g is the sawtooth function.

Having proved a Jackson Theorem in $C^1_{2\pi}$, we can use a bootstrap argument to show that if f is smoother, then the rate of convergence of the best approximation is better. This is the Jackson Theorem in $C^k_{2\pi}$.

Theorem

If $f \in C^k_{2\pi}$, some $k > 0$, then

$$\inf_{p \in \mathcal{T}_n} \|f - p\| \le \left[\frac{\pi}{2(n+1)}\right]^k \left\|f^{(k)}\right\|$$

We shall use induction on k, the case $k = 1$ having been established. Assuming the result, we must show it holds when k is replaced by $k + 1$.

Now let $q \in \mathcal{T}_n$ be arbitrary. Then

$$\inf_{p \in \mathcal{T}_n} \|f - p\| = \inf_{p \in \mathcal{T}_n} \|f - q - p\| \leq \left[\frac{\pi}{2(n+1)}\right]^k \left\|(f-q)^{(k)}\right\|$$

by the inductive hypothesis. Since q is arbitrary and $\left\{p^{(k)} \mid p \in \mathcal{T}_n\right\} = \hat{\mathcal{T}}_n$,

$$\begin{aligned}\inf_{p \in \mathcal{T}_n} \|f - p\| &\leq \left[\frac{\pi}{2(n+1)}\right]^k \inf_{r \in \hat{\mathcal{T}}_n} \left\|f^{(k)} - r\right\| \\ &\leq \left[\frac{\pi}{2(n+1)}\right]^k \frac{\pi}{2(n+1)} \left\|f^{(k+1)}\right\|\end{aligned}$$

To obtain the Jackson theorems for algebraic polynomials we use the following transformation. Given $f : [-1, 1] \to \mathbb{R}$ define $g : \mathbb{R} \to \mathbb{R}$ by $g(\theta) = f(\cos\theta)$. Then g is 2π-periodic and even. This transformation is a linear isometry ($\|g\| = \|f\|$). Note that if $f \in C^1([-1, 1])$ then $g \in C^1_{2\pi}$ and $g'(\theta) = -f'(\cos\theta)\sin\theta$, so $\|g'\| \leq \|f'\|$. Also if $f(x) = x^n$, then $g(\theta) = \left[\left(e^{ix} + e^{-ix}\right)/2\right]^n$ which is a trigonometric polynomial of degree at most n. Thus this transformation maps $\mathcal{P}_n([-1, 1])$ to the $\mathcal{T}_n^{\text{even}}$, the subspace of even functions in $\mathcal{T}_n$, or, equivalently, the span of $\cos kx, k = 0, \ldots, n$. Since $\dim \mathcal{T}_n^{\text{even}} = \dim \mathcal{P}_n([-1, 1]) = n + 1$, the transformation is in fact an isomorphism.

The Jackson theorem in $C^1([-1,1])$ follows immediately from that in $C^1_{2\pi}$:

Theorem

If $f \in C^1([-1,1])$, then

$$\inf_{p \in \mathcal{P}_n} \|f - p\| \leq \frac{\pi}{2(n+1)} \left\|f'\right\|.$$

The Jackson theorem in C^k :

Theorem

Let k be a positive integer, $n \geq k-1$ an integer. Then there exists a constant c depending only on k such that

$$\inf_{p \in \mathcal{P}_n} \|f - p\| \leq \frac{c}{n^k} \left\|f^{(k)}\right\|.$$

for all $f \in C^k([-1,1])$