Lecture 3: Characterization of the minimax approximant

Pr. Ismail Merabet

Univ. of K-M-Ouargla

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Contents

Introduction

We recall that we are interested on the following approximation problem: For a given $f \in C[a, b]$,

$$\begin{cases} \text{find } p_n \in \mathcal{P}_n \text{ such that } : \\ \|f - p_n\|_\infty = \inf_{q \in \mathcal{P}_n} \|f - q\|_\infty \end{cases} \tag{1}$$

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$$\|f - p_0\|_\infty = \max_{x \in [0,1]} |f(x) - c_0| \qquad \text{is minimal.}$$

Since f is monotonic, increasing $f(x) - c_0$ attains :

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Since f is monotonic, increasing $f(x) - c_0$ attains :

- its minimum at $x = 0$ and its maximum at $x = 1$;

therefore, $|f(x) - c_0|$ reaches its maximum value at one of the end-points, i.e.

$$E(c_0) = \max_{x\in[0,1]} |f(x) - c_0| = \max\{|c_0 - f(0)|, |f(1) - c_0|\}$$

Clearly,

$$E(c_0) = \begin{cases} f(1) - c_0 & \text{if } c_0 < \frac{1}{2}[f(0) + f(1)] \\ c_0 - f(0) & \text{if } c_0 \geq \frac{1}{2}[f(0) + f(1)] \end{cases}$$

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Furthermore,

$$\|f-p_0\|_\infty=\frac{1}{2}[f(1)-f(0)].$$

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We shall prove that a property of this kind holds in general:
the precise formulation of the general result will be given later, which is, due to the oscillating nature of the approximation error, usually referred to as **the Oscillation Theorem**.

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This theorem gives:

- a complete characterization of the minimax polynomial and
- provides a method for its construction.

Let us first begin with the following preliminary result:

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- Given $(n+2)$ points $x_0 < \ldots < x_{n+1}$ in the interval $[a, b]$.
- Suppose that

$$\text{sign}\left\{\left[f\left(x_i\right) - r\left(x_i\right)\right]\left(-1\right)^i, \quad i = 0, \ldots, n+1\right\} = \text{ constant} \qquad (**)$$

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Then

$$\min_{q \in \mathcal{P}_n} \|f - q\|_\infty \geq \mu := \min_{i=0,\ldots,n+1} |f(x_i) - r(x_i)|. \qquad (2)$$

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(**) means:

- in passing from a point x_i to the next point x_{i+1} the quantity $f(x) - r(x)$ changes sign (n+2) times .

Proof

For the case $\mu = 0$ the assertion of the theorem is obvious, so let us assume that $\mu > 0$. Suppose that (2) is not true; then, for the minimax polynomial approximation $p_n \in \mathcal{P}_n$ to the function f, we have that

$$\|f - p_n\|_\infty = \min_{q \in \mathcal{P}_n} \|f - q\|_\infty < \mu$$

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$$\operatorname{sign}[r(x) - p_n(x)] = \operatorname{sign}\{[r(x) - f(x)] - [p_n(x) - f(x)]\}$$

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The Oscillation Theorem

Theorem

Suppose that $f \in C[a, b]$. For $r \in \mathcal{P}_n$ to be a minimax polynomial approximation to f over $[a, b]$ it is necessary and sufficient that there exists a sequence of $(n+2)$ points x_j, where

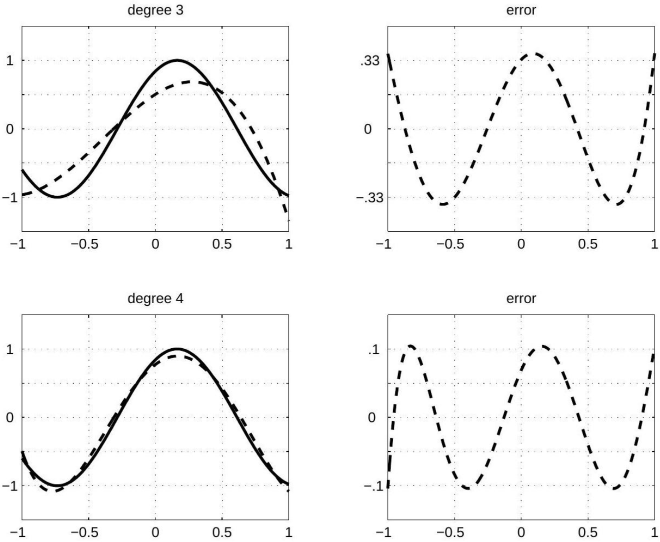
$$a \leq x_0 \leq \ldots \leq x_{n+1} \leq b,$$

such that

$$f(x_i) - r(x_i) = \pm \|f - r\|_\infty, \quad i = 0, \ldots, n+1$$

The points $x_0, \ldots, x_{n+1}$ which satisfy the conditions of the theorem are called critical points.

Illustration



Proof:

Sufficiency. Let L denote the quantity $\|f - r\|_\infty$, and define

$$E_n(f) = \min_{q \in \mathcal{P}_n} \|f - q\|_\infty.$$

Applying (2) it follows that $L = \mu \leq E_n(f)$.

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However, by the definition of $E_n(f)$ we also have that

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Hence $E_n(f) = L$, and the given polynomial r is a minimax polynomial.

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Because $f - r$ is a continuous function on $[a, b]$, we have

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Let us denote by y_2 the lower bound of all points $x \in (y_1, b]$ at which $f(x) - r(x) = -L$, and denote, in succession, by y_{k+1} the lower bound of points $x \in (y_k, b]$ at which $f(x) - r(x) = (-1)^k L$, etc.

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$$f(y_{k+1}) - r(y_{k+1}) = (-1)^k L$$

Let us continue this process up to $y_m = b$, or y_m such that

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We define $z_0 = a$ and $z_m = b$. According to our construction, there exist points in the intervals $[z_{i-1}, z_i]$, $i = 1, \ldots, m$, at which

$$f(x) - r(x) = (-1)^{i-1} L$$

(such are the points y_i, for example), and there is no point x in $[z_{i-1}, z_i]$ where

$$f(x) - r(x) = (-1)^{i} L.$$

We define

$$v(x) = \prod_{j=1}^{m-1} (z_j - x), \qquad \text{and} \qquad r(x;\epsilon) = r(x) + \epsilon v(x), \epsilon > 0$$

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on the intervals $[z_{j-1}, z_j]$. Take, for example, the interval $[z_0, z_1]$. On $[z_0, z_1)$ we have $v(x) > 0$ and therefore

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On $[z_0, z_1)$ we have $v(x) > 0$ and therefore

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At the same time, $f(x) - r(x) > -L$ on $[z_0, z_1]$, and so for ϵ sufficiently small, say, for

$$\epsilon < \epsilon_1 = \frac{\min_{x\in[z_0,z_1]} |f(x) - r(x) + L|}{\max_{x\in[z_0,z_1]} |v(x)|}$$

we have that $f(x) - r(x;\epsilon) > -L$ for all x in $[z_0, z_1)$.

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for ϵ sufficiently small. Arguing in the same manner on the other intervals $[z_{j-1}, z_j]$, $j = 2, \ldots, m$, we can choose ϵ_0 such that

$$|f(x) - r(x;\epsilon_0)| < L, \qquad \text{for } x \in \bigcup_{j=1}^{m} [z_{j-1}, z_j] = [a, b].$$

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Since, by hypothesis, $m < n + 2$, it follows that $m - 1 < n + 1$, and therefore $v \in \mathcal{P}_n$; consequently, $r(x;\epsilon_0) \in \mathcal{P}_n$.

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for ϵ sufficiently small. Arguing in the same manner on the other intervals $[z_{j-1}, z_j]$, $j = 2, \ldots, m$, we can choose ϵ_0 such that

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Since, by hypothesis, $m < n + 2$, it follows that $m - 1 < n + 1$, and therefore $v \in \mathcal{P}_n$; consequently, $r(x;\epsilon_0) \in \mathcal{P}_n$. Thus we have constructed a polynomial $r(x;\epsilon_0) \in \mathcal{P}_n$ such that

$$\|f - r(\cdot;\epsilon_0)\|_\infty < L = \|f - r\|_\infty$$

which contradicts the assumption that r is a polynomial of best approximation to the function f on the interval $[a, b]$ and, simultaneously, $m < n + 2$.

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$$\begin{aligned} f(a) - (c_1 a + c_0) &= \alpha L, \\ f(d) - (c_1 d + c_0) &= -\alpha L, \\ f(b) - (c_1 b + c_0) &= \alpha L \end{aligned}$$

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Thus,

$$c_0 = -\alpha L$$

$$c_0 = \frac{1}{2}\left(\tan^{-1} d - c_1 d\right)$$

$$c_1 = \tan^{-1} 1$$

$$c_1 = \frac{1}{1 + d^2}$$