Lecture 4: Best approximation in the L^2 norm

Pr. Ismail Merabet

Univ. of K-M-Ouargla

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Contents

1. Introduction
2. A simple example
3. Orthogonal polynomials
4. Examples of orthogonal polynomials

Introduction

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$$L_w^2(a, b) := \left\{ f :]a, b[\to \mathbb{R}; \quad \|f\|_2 = \left(\int_a^b w(x) |f(x)|^2 \, dx \right)^{1/2} < \infty \right\}$$

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The problem of best approximation in the $L2$-norm reads:
Given that $f \in L^2_w(a,b)$,

$$\begin{cases} \text{find } p_n \in \mathcal{P}_n \text{ such that} \\ \|f - p_n\|_2 = \inf\limits_{q \in \mathcal{P}_n} \|f - q\|_2 \end{cases} \tag{1}$$

p_n is called a polynomial of best approximation of degree n to the function f in the $L2$-norm.

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Theorem 1.

Given that $f \in L^2_w(a, b)$, there exists a unique polynomial $p_n \in \mathcal{P}_n$ such that

$$\|f - p_n\|_2 = \inf_{q \in \mathcal{P}_n} \|f - q\|_2. \tag{P}$$

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$$p_n(x) = c_n x^n + \ldots + c_1 x + c_0,$$

we want to choose the coefficients c_j so as to minimise the 2-norm of the error, $e_n = f - p_n$,

$$\|e_n\|_2 = \|f - p_n\|_2 = \left(\int_0^1 |f(x) - p_n(x)|^2 \, \mathrm{d}x \right)^{1/2}$$

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$$E(c_0, c_1, \ldots, c_n) = \int_0^1 [f(x) - p_n(x)]^2 \, \mathrm{d}x$$

$$= \int_0^1 [f(x)]^2 \, \mathrm{d}x - 2\sum_{j=0}^{n} c_j \int_0^1 f(x)x^j \, \mathrm{d}x + \sum_{j=0}^{n}\sum_{k=0}^{n} c_j c_k \int_0^1 x^{j+k} \, \mathrm{d}x.$$

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At the unique minimum, the partial derivatives of E with respect to the $c_j, j = 0, \ldots, n$, are equal to zero. This leads to a system of $(n+1)$ linear equations for the coefficients $c_0, \ldots, c_n$:

$$\sum_{k=0}^{n} M_{jk} c_k = b_j, \quad j = 0, \ldots, n \tag{2}$$

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where

$$M_{jk} = \int_0^1 x^{j+k} \, \mathrm{d}x = \frac{1}{j+k+1}, \quad b_j = \int_0^1 f(x)x^j \, \mathrm{d}x.$$

Equivalently, recalling that the inner product associated with the 2-norm (in the case of $w(x) \equiv 1$) is defined by

$$(g, h) = \int_0^1 g(x)h(x)\mathrm{d}x$$

M_{jk} and b_j can be written as

$$M_{jk} = \left(x^k, x^j\right), \quad b_j = \left(f, x^j\right) \tag{3}$$

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By solving the system of linear equations (2), we obtain the coefficients of the polynomial of best approximation of degree n to the function f in the 2-norm on the interval $[0, 1]$. We can proceed in the same manner on any interval $[a, b]$ and any positive weight-function w.

Example

Example 1.

Given that $f(x) = x^2$ for $x \in [0,1]$, find the polynomial p_1 of degree 1 of best approximation to f in the 2 -norm on the interval $[0,1]$ assuming that the weight function is $w(x) \equiv 1$.

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We seek $p_1(x) = c_1 x + c_0$ such that

$$E(c_0, c_1) = \int_0^1 \left[x^2 - (c_1 x + c_0)\right]^2 \, \mathrm{d}x \to \text{ minimum.}$$

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At the minimum, we have that $\frac{\partial E}{\partial c_0} = 0$ and $\frac{\partial E}{\partial c_1} = 0$;therefore,

$$\int_0^1 2\left(x^2 - \left(c_1 x + c_0\right)\right) \cdot (-1) \mathrm{d}x = 0$$
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Solving this gives $c_0 = -\frac{1}{6}$ and $c_1 = 1$ and therefore,

$$p_1(x) = x - \frac{1}{6}$$

Is the required polynomial of best approximation.

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Returning to the general discussion concerning the solution of the linear system (2), we see that we have to invert the matrix $M = (M_{jk})$ with $(n+1)$ rows and columns;

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The matrix $M = (M_{jk})$ with $M_{jk} = 1/(j + k + 1)$ is called the Hilbert matrix. It isn't easy to invert because it is close to being singular. For example, for $n = 10$ when the matrix is of size 11×11, the smallest eigenvalue is approximately 1.9×10^{-13}.

Orthogonal polynomials

In the previous section, we described a method for constructing the polynomial of best approximation $p_n \in \mathcal{P}_n$ to a function f in the 2-norm; it was based on seeking p_n as a linear combination of the polynomials

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Repeating the same process as in the previous section, we arrive at a system of linear equations of the form:

$$\sum_{k=0}^{n} M_{jk}\gamma_k = \beta_j, \quad j = 0, \ldots, n$$

where now

$$M_{jk} = (\phi_k, \phi_j), \quad \text{and} \quad \beta_j = (f, \phi_j).$$

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Thus, $M = (M_{jk})$ will be a diagonal matrix provided the basis functions $\phi_j(x)$, $j = 0, \ldots, n$, for the space $\mathcal{P}_n$ are chosen so that $(\phi_k, \phi_j) = 0$, for $j \neq k$;

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Thus, $M = (M_{jk})$ will be a diagonal matrix provided the basis functions $\phi_j(x)$, $j = 0, \ldots, n$, for the space $\mathcal{P}_n$ are chosen so that $(\phi_k, \phi_j) = 0$, for $j \neq k$; in other words, ϕ_k is required to be orthogonal to ϕ_j for $j \neq k$.

Definition 1.

Given a weight function w, defined and continuous on the interval $[a, b]$ and positive on (a, b), we say that the sequence of polynomials $\phi_j(x), j = 0, 1, \ldots$, forms a system of orthogonal polynomials on the interval (a, b) with respect to w, if each $\phi_j(x)$ is of exact degree j, and if

$$\begin{aligned} \int_a^b w(x)\phi_j(x)\phi_k(x)dx &= 0 \quad \text{for all } j \neq k, \\ &\neq 0 \quad \text{when } j = k. \end{aligned}$$

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We show that a sequence of orthogonal polynomials exists on any interval (a, b) and for any weight function w that is continuous on $[a, b]$ and positive on (a, b), by providing a method of construction.

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We show that a sequence of orthogonal polynomials exists on any interval (a, b) and for any weight function w that is continuous on $[a, b]$ and positive on (a, b), by providing a method of construction.
Let $\phi_0(x) \equiv 1$, and suppose that $\phi_j(x)$ has already been constructed for $j = 0, \ldots, n$. Then

$$\int_a^b w(x)\phi_j(x)\phi_k(x)\mathrm{d}x = 0, \quad 0 \leq j < k \leq n$$

Now let us define the polynomial

$$q(x) = x^{n+1} - a_0\phi_0(x) - \ldots - a_n\phi_n(x)$$

where

$$a_j = \frac{\int_a^b w(x) x^{n+1} \phi_j(x) \mathrm{d}x}{\int_a^b w(x) \left[\phi_j(x)\right]^2 \ \mathrm{d}x}$$

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Then it follows that

$$\begin{aligned}\int_a^b w(x)q(x)\phi_j(x)\mathrm{d}x = &\int_a^b w(x)x^{n+1}\phi_j(x)\mathrm{d}x - a_j\int_a^b w(x)\left[\phi_j(x)\right]^2 \ \mathrm{d}x\\ =&0 \quad \text{for } 0 \le j \le n\end{aligned}$$

Where we have used the orthogonality of the sequence for $j = 0, 1, \ldots, n$.

Thus, with this choice of the numbers a_j, we have ensured that $q(x)$ is orthogonal to all the previous members of the sequence, and $\phi_{n+1}(x)$ can now be defined as any non-zero constant multiple of $q(x)$.

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Example 2.

Construct a system of orthogonal polynomials $\{\psi_0, \psi_1, \psi_2\}$ on the interval $(0, 1)$ concerning the weight function $w(x) \equiv 1$.

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Example 2.

Construct a system of orthogonal polynomials $\{\psi_0, \psi_1, \psi_2\}$ on the interval $(0, 1)$ concerning the weight function $w(x) \equiv 1$.

Given that $w(x) \equiv 1$, the inner product of $L^2_w(0, 1)$ is defined by

$$(u, v) = \int_0^1 u(x)v(x)\mathrm{d}x$$

We put $\psi_0(x) \equiv 1$, and we seek ψ_1 in the form

$$\psi_1(x) = x - c_0 \psi_0(x)$$

such that $(\psi_1, \psi_0) = 0$; namely,

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Hence,

$$c_0 = \frac{(x, \psi_0)}{(\psi_0, \psi_0)} = \frac{1/2}{1} = \frac{1}{2}$$

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$$\psi_1(x) = x - \frac{1}{2}\psi_0(x) = x - \frac{1}{2}$$

By construction, $(\psi_1, \psi_0) = (\psi_0, \psi_1) = 0$.

Now we seek ψ_2 in the form

$$\psi_2(x) = x^2 - (d_1\psi_1(x) + d_0\psi_0(x))$$

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$$\begin{aligned}
\left(x^2, \psi_1\right) - d_1\left(\psi_1, \psi_1\right) - d_0\left(\psi_0, \psi_1\right) = 0, \\
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As $(\psi_0, \psi_1) = 0$ and $(\psi_1, \psi_0) = 0$, we have that

$$\begin{aligned} d_1 &= \frac{(x^2, \psi_1)}{(\psi_1, \psi_1)} = 1 \\ d_0 &= \frac{(x^2, \psi_0)}{(\psi_0, \psi_0)} = \frac{1}{3} \end{aligned}$$

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$$\psi_2(x) = x^2 - (d_1\psi_1(x) + d_0\psi_0(x))$$

such that $(\psi_2, \psi_1) = 0$ and $(\psi_2, \psi_0) = 0$. Thus,

$$\left(x^2, \psi_1\right) - d_1\left(\psi_1, \psi_1\right) - d_0\left(\psi_0, \psi_1\right) = 0,$$
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As $(\psi_0, \psi_1) = 0$ and $(\psi_1, \psi_0) = 0$, we have that

$$d_1 = \frac{\left(x^2, \psi_1\right)}{(\psi_1, \psi_1)} = 1$$
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and therefore

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Clearly, $(\psi_j, \psi_k) = 0$ for $j \neq k, j, k \in \{0, 1, 2\}$, and ψ_k is of exact degree k, $k = 0, 1, 2$, so we have found the required system of orthogonal polynomials on the interval $(0, 1)$. $\diamond$

Legendre polynomials

The polynomials

$$\phi_0(x) = 1,$$
$$\phi_1(x) = x,$$
$$\phi_2(x) = x^2 - \frac{1}{3}$$
$$\phi_3(x) = x^3 - \frac{3}{5}x$$

Are the first four elements of an orthogonal system on the interval $(-1, 1)$ concerning the weight function $w(x) \equiv 1$.

Chebyshev polynomials

The polynomials

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x, \\
T_4(x) &= 8x^4 - 8x^2 + 1, \\
T_5(x) &= 16x^5 - 20x^3 + 5x, \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1
\end{aligned}$$

are the first seven elements of an orthogonal system on the interval $(-1, 1)$ with respect to the positive weight function $w(x) = \left(1 - x^2\right)^{-1/2}, x \in (-1, 1)$. (This weight function is continuous on $(-1, 1)$, but not on $[-1, 1]$).

The next theorem, in conjunction with the use of orthogonal polynomials, is the key tool for constructing the polynomial of best approximation in the 2-norm.

Theorem 2.

The polynomial $p_n \in \mathcal{P}_n$ is the polynomial of best approximation of degree n to a function $f \in L^2_w(a,b)$ in the 2-norm if and only if the difference $f - p_n$ is orthogonal to every element of $\mathcal{P}_n$, i.e.

$$\int_a^b w(x)\,(f(x) - p_n(x))\,q(x)dx = 0 \qquad \text{for all } q \in \mathcal{P}_n$$

Theorem 2 provides a simple method of determining the polynomial of best approximation p_n to a function $f \in L^2_w(a,b)$ in the 2-norm.

- First, we construct the system of orthogonal polynomials $\phi_j(x), j = 0, \ldots, n$, on the interval (a, b) with respect to the weight function w, if this system is not already known.

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$$\int_a^b w(x)\left[f(x) - \sum_{k=0}^{n}\gamma_k\phi_k(x)\right]\phi_j(x)\mathrm{d}x = 0, \quad j = 0, 1, \ldots, n$$

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Exploiting the orthogonality of the polynomials ϕ_j, this gives

$$\gamma_j = \frac{\int_a^b w(x)f(x)\phi_j(x)\mathrm{d}x}{\int_a^b w(x)\left[\phi_j(x)\right]^2 \, \mathrm{d}x} \left(= \frac{(f, \phi_j)}{\|\phi_j\|_2^2} \right)$$

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Construct the polynomial of best approximation of degree 2 in the $L2$-norm to the function $f : x \mapsto \mathrm{e}^x$ over $[-1, 1]$ with weight function $w(x) \equiv 1$.

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We already know a set of orthogonal polynomials ϕ_0, ϕ_1, ϕ_2 on this interval; thus we seek $p_2 \in \mathcal{P}_2$ in the form

$$p_2(x) = \gamma_0\phi_0(x) + \gamma_1\phi_1(x) + \gamma_2\phi_2(x) \tag{4}$$

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Requiring that

$$f(x) - p_2(x) = e^x - (\gamma_0\phi_0(x) + \gamma_1\phi_1(x) + \gamma_2\phi_2(x))$$

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be orthogonal to ϕ_0, ϕ_1 and ϕ_2, i.e. that

$$\int_{-1}^{1} \left[e^x - (\gamma_0\phi_0(x) + \gamma_1\phi_1(x) + \gamma_2\phi_2(x))\right] \phi_j(x)\mathrm{d}x = 0, \quad j = 0, 1, 2$$

we obtain the coefficients $\gamma_j, j = 0, 1, 2$:

$$\gamma_0 = \frac{\int_{-1}^{1} e^x \, \mathrm{d}x}{2} = \frac{e - 1/e}{2}$$

$$\gamma_1 = \frac{\int_{-1}^{1} e^x x \, \mathrm{d}x}{2/3} = \frac{3}{e}$$

$$\gamma_2 = \frac{\int_{-1}^{1} e^x \left(x - \frac{1}{3}\right)^2 \, \mathrm{d}x}{8/45} = \frac{45}{8}\left(\frac{2e}{3} - \frac{14}{3e}\right)$$

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Substituting the values of γ_0, γ_1 and γ_2 into (4) and recalling the expressions for $\phi_0(x)$, $\phi_1(x)$ and $\phi_2(x)$, we obtain the polynomial of best approximation of degree 2 for the function f. $\diamond$