Lecture 5: Noncoercive problems: Babuška theorem

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Contents

Introduction

So far we have treated coercive problems. This means that in the linear variational problem, we are trying to solve,

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v) = F(v), \text{ for all } v \in V \end{cases} \tag{1}$$

the bilinear form $a(u,v)$ satisfies

$$a(v,v) \geq \alpha \|v\|_V^2, \quad \forall v \in V$$

for some $\alpha > 0$.
Recall that the best constant α satisfying the definition is given by

$$\alpha := \inf_{\substack{v \in V \\ v \neq 0}} \frac{a(v,v)}{\|v\|_V^2}$$

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We now consider noncoercive problems, one for which no such $\alpha > 0$ exists. We will develop more general (necessary and sufficient) criteria for the well-posedness of the linear variational problem, known as the inf-sup or Babuška conditions.

For coercive problems, well-posedness is inherited for $V_h \subset V$. This is not true for noncoercive problems. Well-posedness is not inherited for an arbitrary $V_h \subset V$. One must prove the stability of each candidate discretisation individually.

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Helmholtz equation

As an example of non-coercive, we can show that the Helmholtz equation

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\begin{aligned}
-\Delta u - k^2 u &= f \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned}
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is well-posed if k^2 is not an eigenvalue of the Dirichlet Laplacian, but is not coercive for k large enough. For k^2 to be an eigenvalue of the Dirichlet Laplacian, it means that there exists $u \neq 0$ such that

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Mixed Laplacian

Suppose we want to accurately determine the flux in the Poisson equation.

We can solve the mixed formulation:

find $\sigma : \Omega \to \mathbb{R}^n, u : \Omega \to \mathbb{R}$ such that

$$\begin{aligned} \sigma &= -\nabla u && \text{in } \Omega, \\ \mathrm{div}\sigma &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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mixed Laplacian

Let's multiply the first equation by a vector-valued test function τ, and the second by a scalar-valued function v:

$$\int_\Omega \sigma \cdot \tau \, \mathrm{d}x + \int_\Omega \nabla u \cdot \tau = 0$$
$$\int_\Omega \mathrm{div}\,(\sigma) v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x$$

Since σ needs to have a divergence, and we want τ and σ to come from the same space, let's integrate by parts in the first equation. For symmetry, I'll negate the second equation:

$$\int_\Omega \sigma \cdot \tau \, \mathrm{d}x - \int_\Omega u \, \mathrm{div}(\tau) + \int_{\partial\Omega} u \tau \cdot n \, \mathrm{d}s = 0$$
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We don't need any derivatives on u or v, so $u \in L^2(\Omega)$. For σ and τ, we need $\sigma \in L^2(\Omega; \mathbb{R}^n)$ and for div $\sigma \in L^2(\Omega)$. This is the space $H(\text{div}, \Omega)$:

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A nice property of variational problems is that we can add the two equations together. The problem is the same as:

$$\begin{cases} \text{Find } (\sigma, u) \in H(\text{ div }, \Omega) \times L^2(\Omega) \text{ such that} \\ \int_\Omega \sigma \cdot \tau \, \mathrm{d}x - \int_\Omega \text{ div}(\tau) u - \int_\Omega \text{ div}(\sigma) v \, \mathrm{d}x = - \int_\Omega f v \, \mathrm{d}x \end{cases}$$

for all $(\tau, v) \in H(\text{div}, \Omega) \times L^2(\Omega)$.

We define:

$$B(\sigma, u; \tau, v) := \int_\Omega \sigma \cdot \tau \, \mathrm{d}x - \int_\Omega \text{ div}(\tau) u - \int_\Omega \text{ div}(\sigma) v \, \mathrm{d}x$$

Lax-Milgram certainly won't apply:

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Theorem (Babuška theorem)

Let V and W be two Hilbert spaces. Let $a : V \times W \to \mathbb{R}$ be a bilinear form for which there exist constants $C < \infty, \gamma > 0, \gamma' > 0$ such that:

1. $|a(v, w)| \leq C\|v\|_V \|w\|_W, \quad$ for all $v \in V, w \in W$,
2. $$\inf_{\substack{v \in V \\ v \neq 0}} \sup_{\substack{w \in W \\ w \neq 0}} \frac{a(v, w)}{\|v\|_V \|w\|_W} \geq \gamma > 0,$$
3. $$\inf_{\substack{w \in W \\ w \neq 0}} \sup_{\substack{v \in V \\ v \neq 0}} \frac{a(v, w)}{\|v\|_V \|w\|_W} \geq \gamma' > 0.$$

Then for any $F \in W'$ there exists exactly one solution $u \in V$ such that:

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Furthermore, the problem is stable, i.e,

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As a first example of how to manipulate inf-sup conditions, let's show that a coercive problem satisfies the inf-sup conditions. Suppose $a(u, v)$ satisfies

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Dividing both sides of the inequality by $\|u\|_V$ for $u \neq 0$, we have

$$\begin{aligned} \alpha \|u\|_V &\le \frac{a(u, u)}{\|u\|_V} \\ &\le \sup_{\substack{v \in V \\ v \neq 0}} \frac{a(u, v)}{\|v\|_V} \end{aligned}$$

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and take the Galerkin approximation over closed $V_h \subset V$:
find $u_h \in V_h$ such that

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Note that Galerkin orthogonality still holds.

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Let's check the Babuška conditions.

1. Satisfaction of (1) is inherited.
2. What about (2)?

That is, does there exist $\tilde{\gamma}$ such that

$$\inf_{\substack{u_h \in V_h \\ u_h \neq 0}} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{a(u_h, v_h)}{\|u_h\|_V \|v_h\|_V} \geq \tilde{\gamma} > 0$$

with $\tilde{\gamma}$ independent of the mesh size h ?No! Examples later.

Remark

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We don't need to check (3) in this case! The discrete system is square and finite-dimensional, so (2) $\iff$ (3) by rank-nullity.)

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That is, does there exist $\tilde{\gamma}$ such that

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Quasi-optimality

Theorem

Assume we have a well-posed discretisation of a well-posed problem. Then

$$\|u - u_h\|_V \leq \left(1 + \frac{C}{\tilde{\gamma}}\right) \|u - v_h\|_V$$

Proof

For every $v_h \in V_h$, we have

$$\begin{aligned}
\tilde{\gamma} \left\| v_h - u_h \right\|_V &\leq \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{a\left(v_h - u_h, w_h\right)}{\left\| w_h \right\|_V} && \text{(discrete inf-sup)} \\
&= \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{a\left(v_h - u, w_h\right) + a\left(u - u_h, w_h\right)}{\left\| w_h \right\|_V} \\
&= \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{a\left(v_h - u, w_h\right)}{\left\| w_h \right\|_V} && \text{(Galerkin orth.)} \\
&\leq \sup_{\substack{w_h \in V_h \\ w_h \neq 0}} \frac{C \left\| v_h - u \right\|_V \left\| w_h \right\|_V}{\left\| w_h \right\|_V}
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Now apply the triangle inequality to $\|u - u_h\|_V$:

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