Lecture 6: Saddle point problems
The Brezzi theorem

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Contents

1. Introduction
2. Examples of saddle point problems
3. Energy minimisation
4. Well-posedness of saddle point problems
5. Discretisation of saddle point problems

Introduction

We have now seen the general necessary and sufficient Babuška conditions for the well-posedness of find $u \in V$ such that $a(u, v) = F(v)$ for all $v \in V$.

Many noncoercive problems arise via mixed formulations (solving for more than one variable), and in this lecture we will rephrase the well-posedness conditions for saddle point problems:

$$\begin{cases} \text{find}(u, p) \in V \times Q \text{ such that} \\ a(u, v) + b(v, p) = F(v), \quad \forall v \in V \\ b(u, q) = G(q), \quad \forall q \in Q \end{cases} \tag{1}$$

These are the Brezzi conditions. The Brezzi conditions are easier to understand and verify than the Babuška conditions if you have a saddle point problem.

Note that the problem:

$$\begin{cases} \text{find } (u,p) \in V \times Q \text{ such that} \\ a(u,v) + b(v,p) = F(v), \quad \forall v \in V \\ \qquad\qquad\quad b(u,q) = G(q), \quad \forall q \in Q \end{cases} \tag{2}$$

is equivalent to:

$$\begin{cases} \text{find } (u,p) \in V \times Q \text{ such that} \\ a(u,v) + b(v,p) + b(u,q) = F(v) + G(q), \quad \forall (v,q) \in V \times Q \end{cases} \tag{3}$$

- Set $v = 0$ and vary $q \in Q$,
- set $q = 0$ and vary $v \in V$.

Mixed Poisson

We've already seen one example: Mixed Poisson (lecture 5)

$$\begin{cases} \text{Find } (\sigma, u) \in H(\mathrm{div}, \Omega) \times L^2(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \sigma \cdot v \, \mathrm{d}x - \int_\Omega \mathrm{div}(v) u - \int_\Omega \mathrm{div}(\sigma) w \, \mathrm{d}x = -\int_\Omega f w \, \mathrm{d}x \\ \text{for all } (v, w) \in H(\mathrm{div}, \Omega) \times L^2(\Omega). \end{cases} \tag{4}$$

Here

$$a(\sigma, v) = \int_\Omega \sigma \cdot v \, \mathrm{d}x, \quad b(v, u) = -\int_\Omega \mathrm{div}(v) u \, \mathrm{d}x$$

The Stokes problem

Let's consider one more example. The Stokes equations are an elementary model in fluid mechanics. They describe the motion of a steady, incompressible, viscous, Newtonian, isothermal, slow-moving fluid.

$$
\begin{aligned}
-\Delta u + \nabla p &= f \text{ in } \Omega \\
\operatorname{div} u &= 0 \text{ in } \Omega \\
u &= 0 \text{ on } \partial\Omega
\end{aligned}
$$

Here $u : \Omega \to \mathbb{R}^n$ is the flow velocity and $p : \Omega \to \mathbb{R}$ is the pressure.

The Stokes problem

Multiply the momentum equation by a vector-valued test function $v \in V$, and the continuity equation by a scalar-valued test function $q \in Q$:

$$\int_\Omega - \operatorname{div}\,(\nabla u) \cdot v \;\mathrm{d}x + \int_\Omega \nabla p \cdot v \;\mathrm{d}x = \int_\Omega f \cdot v \;\mathrm{d}x$$
$$\int_\Omega q \operatorname{div}\,(u) \;\mathrm{d}x = 0$$

Integrate the vector Laplacian by parts:

$$\int_\Omega \nabla u : \nabla v \;\mathrm{d}x - \int_{\partial\Omega} n \cdot \nabla u \cdot v \;\mathrm{d}s + \int_\Omega \nabla p \cdot v \;\mathrm{d}x = \int_\Omega f \cdot v \;\mathrm{d}x$$
$$\int_\Omega q \operatorname{div}\, u \;\mathrm{d}x = 0$$

The Stokes problem

Multiply the momentum equation by a vector-valued test function $v \in V$, and the continuity equation by a scalar-valued test function $q \in Q$:

$$\int_\Omega -\operatorname{div} \nabla u \cdot v \, \mathrm{d}x + \int_\Omega \nabla p \cdot v \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x$$
$$\int_\Omega q \operatorname{div} u \, \mathrm{d}x = 0.$$

Integrate the vector Laplacian by parts:

$$\int_\Omega \nabla u : \nabla v \, \mathrm{d}x - \int_{\partial\Omega} n \cdot \nabla u \cdot v \, \mathrm{d}s + \int_\Omega \nabla p \cdot v \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x,$$
$$\int_\Omega q \operatorname{div} u \, \mathrm{d}x = 0.$$

We have nowhere to weakly enforce $u = 0$, so take $V = H_0^1(\Omega; \mathbb{R}^n)$.

The Stokes problem

The formulation

$$\int_\Omega \nabla u : \nabla v \, \mathrm{d}x + \int_\Omega \nabla p \cdot v \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x$$
$$\int_\Omega q \, \mathrm{div} \, u \, \mathrm{d}x = 0$$

requires $u \in H_0^1(\Omega; \mathbb{R}^n)$ and $p \in H^1(\Omega)$. We can weaken the regularity requirement to $p \in L^2(\Omega)$ by integrating by parts, and then negating the second equation for symmetry:

$$\int_\Omega \nabla u : \nabla v \, \mathrm{d}x - \int_\Omega p \, \mathrm{div} \, v \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x$$
$$-\int_\Omega q \, \mathrm{div} \, u \, \mathrm{d}x = 0$$

Here

$$a(u, v) = \int_\Omega \nabla u : \nabla v \, \mathrm{d}x, \quad b(v, p) = -\int_\Omega p \, \mathrm{div} \, v \, \mathrm{d}x$$

The Stokes problem

In the strong form of the problem, p only appears via ∇p. So if (u, p) is a solution, so is $(u, p + c)$ for $c \in \mathbb{R}$. We can see this variationally:

$$\begin{aligned}
\int_\Omega (p + c) \text{ div } v \, \mathrm{d}x &= \int_\Omega p \text{ div } v \, \mathrm{d}x + c \int_\Omega \text{ div } v \, \mathrm{d}x \\
&= \int_\Omega p \text{ div } v \, \mathrm{d}x + c \int_{\partial\Omega} v \cdot n \, \mathrm{d}s \\
&= \int_\Omega p \text{ div } v \, \mathrm{d}x.
\end{aligned}$$

To fix a unique pressure we choose

$$Q = L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0 \right\}.$$

Energy minimisation

Consider

$$\begin{aligned} u = \underset{v \in H^1_0(\Omega;\mathbb{R}^n)}{\operatorname{argmin}} \quad & \tfrac{1}{2}\int_\Omega \nabla v : \nabla v \,\mathrm{d}x - \int_\Omega f \cdot v \,\mathrm{d}x, \\ \text{subject to} \quad & \operatorname{div} v = 0 \end{aligned}$$

We introduce a Lagrange multiplier p and write the Lagrangian

$$\begin{aligned} L &: H^1_0(\Omega;\mathbb{R}^n) \times L^2_0(\Omega) \to \mathbb{R} : \\ & L(u,p) = \frac{1}{2}\int_\Omega \nabla u : \nabla u \,\mathrm{d}x - \int_\Omega f \cdot u \,\mathrm{d}x - \int_\Omega p \operatorname{div} u \,\mathrm{d}x. \end{aligned}$$

Euler-Lagrange

$$L(u,p) = \frac{1}{2}\int_\Omega \nabla u : \nabla u \,\mathrm{d}x - \int_\Omega p \,\mathrm{div}\, u \,\mathrm{d}x - \int_\Omega f \cdot u \,\mathrm{d}x.$$

Calculating the Euler-Lagrange equations, we have

$$\begin{aligned}
L_u(u,p;v) &= \int_\Omega \nabla u : \nabla v \,\mathrm{d}x - \int_\Omega p \,\mathrm{div}\, v \,\mathrm{d}x - \int_\Omega f \cdot v \,\mathrm{d}x = 0,\\
L_p(u,p;q) &= -\int_\Omega q \,\mathrm{div}\, u \,\mathrm{d}x = 0,
\end{aligned}$$

the Stokes equations in weak form.
In general constrained optimisation problems give you saddle point problems, because the constraint equation does not involve the Lagrange multiplier.

Well-posedness of saddle point problems

We now state the Brezzi conditions for the well-posedness of the abstract saddle point problem.

Theorem

Let V and Q be Hilbert spaces. Given $F \in V'$ and $G \in Q'$, we consider the problem:

$$\begin{cases} \text{find } (u,p) \in V \times Q \text{ such that} \\ a(u,v) + b(v,p) = F(v), \quad \forall v, \in V \\ \qquad\qquad b(u,q) = G(q), \quad \forall q \in Q. \end{cases} \tag{5}$$

Let

$$\ker b = \{v \in V : b(v,q) = 0 \text{ for all } q \in Q\}$$

Well-posedness of saddle point problems

Theorem

Suppose that:

1. $a : V \times V \to \mathbb{R}$ and $b : V \times Q \to \mathbb{R}$ are bounded bilinear forms;
2. The variational problem:

$$\begin{cases} \text{find } u \in K \text{ such that} \\ a(u, v) = F(v), \quad \forall v \in \ker b \end{cases} \tag{6}$$

   is well-posed;
3. b satisfies the following inf-sup condition: there exists $\gamma \in \mathbb{R}$ such that

$$0 < \gamma \leq \inf_{\substack{q \in Q \\ q \neq 0}} \sup_{\substack{v \in V \\ v \neq 0}} \frac{b(v, q)}{\|v\|_V \|q\|_Q}.$$

Then there exists a unique pair $(u, p) \in V \times Q$ that solves the variational problem, and the solution is stable with respect to the data F and G.

Take $V_h \times Q_h \subset V \times Q$, and consider:

$$\begin{cases} \text{find } (u_h, p_h) \in V_h \times Q_h \text{ such that} \\ a(u_h, v_h) + b(v_h, p_h) = F(v_h), \quad \forall v_h \in V_h \\ b(u_h, q_h) = G(q_h), \quad \forall q_h \in Q_h \end{cases} \tag{7}$$

For this to be well-posed, Brezzi's conditions require that the LVP involving a is well-posed on the discrete kernel

$$\ker_h = \{v_h \in V_h : b(v_h, q_h) = 0 \text{ for all } q_h \in Q_h\}.$$

Compare with

$$\ker b \cap V_h = \{v_h \in V_h : b(v_h, q) = 0 \text{ for all } q \in Q\}.$$

In general, for $v_h \in V_h$, the property

$$b(v_h, q_h) = 0 \text{ for all } q_h \in Q_h$$

will not imply

$$b(v_h, q) = 0 \text{ for all } q \in Q$$

(It will sometimes, but not always.) So in general $\ker_h \not\subset \ker b$. This means that well-posedness of a on the discrete kernel $\ker_h$ does not necessarily follow automatically from well-posedness of a on the full kernel $\ker b$. One way to look at it: we have a non-conforming discretisation of the kernel problem.

That's one way a discretisation might fail. Any others? Given that b satisfies the inf-sup condition over V and Q, it does not follow that b satisfies the inf-sup condition: there exists $\tilde{\gamma} \in \mathbb{R}$ such that

$$0 < \tilde{\gamma} \leq \inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{b\left(v_h, q_h\right)}{\|v_h\| \, \|q_h\|}$$

We will see this by counterexample (later). So to analyse our discretisation error, we must additionally assume the Brezzi conditions hold for our discrete problem. This is a compatibility condition on the elements we choose for V_h and Q_h : they must work together.

Many interesting problems are of saddle point form:

$$\begin{cases} \text{find}(u,p) \in V \times Q \text{ such that} \\ \qquad a(u,v) + b(v,p) = F(v), \quad \forall v \in V \\ \qquad\qquad b(u,q) = G(q), \quad \forall q \in Q \end{cases} \tag{8}$$

For this to be well-posed, we needed continuity of a and b, and

1. The variational problem:

$$\begin{cases} \text{find } u \in \ker b \text{ such that} \\ a(u,v) = F(v), \quad \forall v \in \ker b \end{cases} \tag{9}$$

   over $\ker b := \{v \in V : b(v,q) = 0 \text{ for all } q \in Q\}$ is well-posed;
2. b satisfies the following inf-sup condition: there exists $\gamma \in \mathbb{R}$ such that

$$0 < \gamma \le \inf_{\substack{q \in Q \\ q \neq 0}} \sup_{\substack{v \in V \\ v \neq 0}} \frac{b(v,q)}{\|v\|_V \|q\|_Q}$$

Consider a Galerkin approximation: find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= F(v_h) \\ b(u_h, q_h) &= G(q_h) \end{aligned}$$

for all $(v_h, q_h) \in V_h \times Q_h$. We similarly require:

1. The variational problem find $u_h \in \ker_h$ such that $a(u_h, v_h) = F(v_h)$ for all $v_h \in \ker_h$ over $\ker_h := \{v_h \in V_h : b(v_h, q_h) = 0 \text{ for all } q_h \in Q_h\}$ is well-posed;
2. $V_h \times Q_h$ satisfies the following inf-sup condition: there exists $\tilde{\gamma} \in \mathbb{R}$ such that

$$0 < \tilde{\gamma} \leq \inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{\substack{v_h \in V_h \\ v_h \neq 0}} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q}$$

In this lecture we apply this theory to the mixed Poisson equation.