MEF II: Lecture 1. Construction of finite element spaces

Pr. Ismail Merabet

Univ. of K-M-Ouargla

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Contents

Introduction

In the previous semester, we have studied variational problems of the form:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = F(v), \forall v \in V \end{cases} \tag{1}$$

where:

- V is an infinit dimonsional Hilbert space,
- $a(\cdot,\cdot)$ is a bilinear form and
- $F(\cdot)$ is a linear form.

We have also seen that the Galerkin method replaces the space V by a finite dimensional space V_h.

We saw Céa's Lemma: for a coercive bounded bilinear form a, the error in Galerkin approximation is bounded by

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{v_h \in V_h} \|u - v_h\|_V.$$

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The finite element method!

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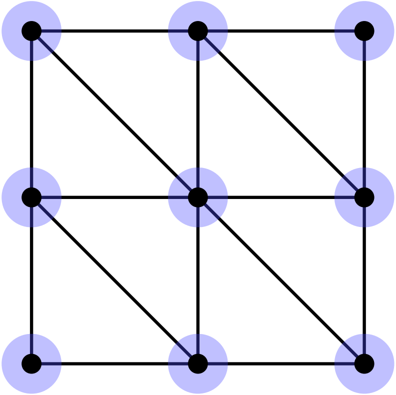
This lecture describes the construction and properties of finite element spaces. We will give examples of:

- conforming finite elements and
- nonconforming finite elements.

Key idea: use piecewise polynomials on a mesh of Ω.



A mesh or triangulation of Ω for piecewise linear function

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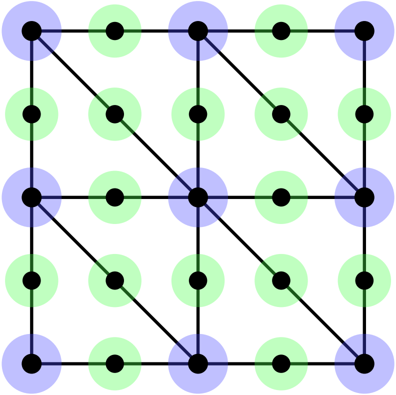

Data stored to represent a piecewise quadratic function

The Construction of a Finite Element Space

Definition 1.2 (Ciarlet 1978)

A finite element is a triple ($K, \mathcal{P}, \mathcal{N}$),

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Usually, condition (iii) of Definition 1.2 is the only one that requires much work.

The following lemma simplifies its verification.

Lemma 1.3

Let $\mathcal{P}$ be a d-dimensional vector space and let $\{N_1, N_2, \ldots, N_d\}$ be a subset of the dual space $\mathcal{P}'$. Then the following two statements are equivalent.

a. $\{N_1, N_2, \ldots, N_d\}$ is a basis for $\mathcal{P}'$.
b. Given $v \in \mathcal{P}$ with $N_i v = 0$ for $i = 1, 2, \ldots, d$, then $v \equiv 0$.

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Definition 1.4

Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element. The basis $\{\phi_1, \phi_2, \ldots, \phi_k\}$ of $\mathcal{P}$ dual to $\mathcal{N}$, i.e

$$N_i(\phi_j) = \delta_{ij}$$

is called the nodal basis of $\mathcal{P}$.

1d-Examples

Example ($\mathbb{P}_1$ Lagrange element).

Let $K = [0,1], \mathcal{P} =$ the set of linear polynomials and $\mathcal{N} = \{N_1, N_2\}$, where

$$N_1(v) = v(0) \quad \text{and} \quad N_2(v) = v(1), \quad \forall v \in \mathcal{P}.$$

Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element and the nodal basis consists of

$$\phi_1(x) = 1 - x \quad \text{and} \quad \phi_2(x) = x.$$

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$$\phi_1(x) = 1 - x \quad \text{and} \quad \phi_2(x) = x.$$

Proof.

Let $v \in \mathbb{P}_1$, means, $v = a + bx$. Then $N_1(v) = N_2(v) = 0$ means

$$\begin{aligned} a &= 0 \\ a + b &= 0. \end{aligned}$$

Hence, $a = b = 0$, i.e., $v \equiv 0$.

$\mathbb{P}_k$ Lagrange element.

Example ($\mathbb{P}_k$ Lagrange element)

In general, we can let $K = [a, b]$ and $\mathcal{P} = \mathbb{P}_k([a, b])$ the set of all polynomials of degree less than or equal to k. Let $\mathcal{N} = \{N_0, N_1, N_2, \ldots, N_k\}$, where

$$N_i(v) = v(a + \frac{(b-a)}{k} i), \quad \forall v \in \mathcal{P} \text{ and } i = 0, 1, \ldots, k.$$

Then $(K, \mathcal{P}, \mathcal{N})$ is a finite element.

Proof.

Example (Hermite finite element).

$K = [0,1], \mathcal{P} = \mathbb{P}_3(K)$, and $\mathcal{N} = \{N_1, N_2, N_3, N_4\}$

$$\begin{aligned} N_1 &: v \mapsto v(0), \\ N_2 &: v \mapsto v'(0), \\ N_3 &: v \mapsto v(1), \\ N_4 &: v \mapsto v'(1). \end{aligned}$$

Proof.

The $\mathbb{P}_1$-Lagrange finite element in two dimensions

The following are examples of two-dimensional finite elements.

Example (Triangular Lagrange Elements)

Let K be a triangle, $\mathcal{P}$ be the space $\mathbb{P}_k$ of polynomials in two variables of degree $\leq k$, and let the set $\mathcal{N}$ consist of evaluations of shape functions at the nodes with barycentric coordinates:

$$z_1 = i/k, \quad z_2 = j/k \quad \text{and} \quad z_3 = \ell/k,$$

where i, j, ℓ are nonnegative integers and $i + j + \ell = k$.
Then $(K, \mathcal{P}, \mathcal{N})$ is the two-dimensional $\mathbb{P}_k$ Lagrange finite element.

The nodal variables for the N_1, N_2, and N_3 Lagrange elements are depicted in Figure 1.

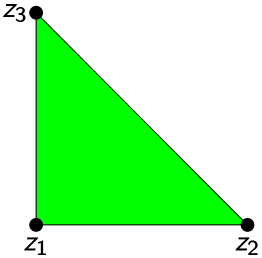


Figure: P1-Lagrange

Here and in the following examples, the symbol • represents pointwise evaluation of shape functions.

$$N_i(v) = v(z_i)$$

Lemma (First factorisation lemma)

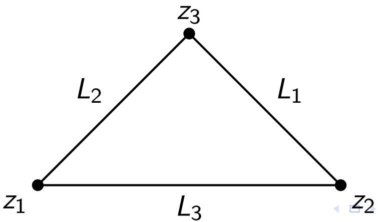
Let P be a polynomial of degree $d \geq 1$ that vanishes on a hyperplane

$$\{x : L(x) = 0\},$$

where $L(x)$ is a non-degenerate linear function.
Then we can write $P = LQ$, where Q is a polynomial of degree $d - 1$.

Example

Suppose P vanishes on each edge of a triangle. Then $P = L_1 L_2 L_3 Q$ for some Q. Here, L_1, L_2 and L_3 are non-trivial linear functions that define the lines on which lie the edges of the triangle.



The $\mathbb{P}_2$-Lagrange finite element in two dimensions

Unisolvence of $\mathbb{P}_2$

Suppose $v \in \mathcal{P}_2(T)$ with all degrees of freedom zero. Restricted to an edge, v is a quadratic polynomial with three roots, hence $v = 0$ on each edge. By the factorization lemma, $v = L_1 L_2 c$ for a constant $c \in \mathbb{R}$. Evaluating both sides on the edge $L_3 = 0$ shows that $c = 0$.

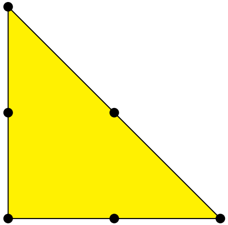


Figure: P2-Lagrange

Lemma (Second factorisation lemma)

Let P be a polynomial of degree $d \geq 2$ such that P and $\nabla P \cdot n$ vanish on a hyperplane $\{x : L(x) = 0\}$, where n is the normal to L. Then we can write $P = L^2 Q$, where Q is a polynomial of degree $d - 2$.

Remark

If P vanishes on $\{x : L(x) = 0\}$, so does $\nabla P \cdot t$ for a tangent vector t.

Proof.

Since P vanishes on $\{x : L(x) = 0\}$, we have $P = L\tilde{Q}$. Calculating,

$$\nabla P \cdot n = \tilde{Q} \nabla L \cdot n + L \nabla \tilde{Q} \cdot n$$

Since L vanishes on the plane, and ∇L is normal to the plane (hence colinear with n), this forces $\tilde{Q} = 0$ on $\{x : L(x) = 0\}$.

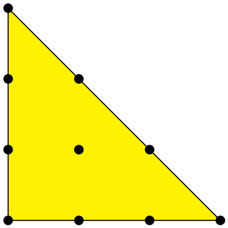


Figure: P3-Lagrange

Let T be a triangle.

Example (Triangular Hermite Elements)

The cubic Hermite element is the triple $(T, \mathbb{P}_3, \mathcal{N})$ where $\mathcal{N}$ consists of evaluations of shape functions and their gradients at the vertices and evaluation of shape functions at the center of T. i.e
$\mathcal{N} = \{N_1, N_2, \ldots, N_{10}\}$,

Here and in the following examples $\odot$ represents pointwise evaluation of gradients of shape functions.

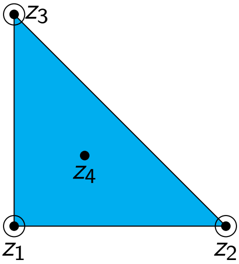


Figure: Cubic Hermite

The triangular Hermite element

Lemma (Unisolvence of the triangular Hermite element).

The Hermite element in two dimensions is unisolvent. i.e $\mathcal{N}$ determine $\mathbb{P}_3$.

Proof.

Let L_1, L_2 and L_3 again be non-trivial linear functions that define the edges of the triangle. Suppose that for a polynomial $P \in \mathcal{P}_3, N_i(P) = 0$ for $i = 1, 2, \ldots, 10$. Restricting P to L_1, we see that z_2 and z_3 are double roots of P since $P(z_2) = 0, P'(z_2) = 0$ and $P(z_3) = 0, P'(z_3) = 0$, where $'$ denotes differentiation along the straight line L_1. But the only third order polynomial in one variable with four roots is the zero polynomial, hence $P \equiv 0$ along L_1. Similarly, $P \equiv 0$ along L_2 and L_3. We can, therefore, write $P = cL_1L_2L_3$. But

$$0 = P(z_4) = cL_1(z_4)\,L_2(z_4)\,L_3(z_4) \quad \Longrightarrow \quad c = 0$$

because $L_i(z_4) \neq 0$ for $i = 1, 2, 3$.

Argyris element

The fifth degree Argyris element is the triple $(K, \mathbb{P}_5, \mathcal{N}_K)$ where $\mathcal{N}_K$ consists of evaluations of the shape functions and their derivatives up to order two at the vertices and evaluations of the normal derivatives at the midpoints of the edges. The nodal variables for the Argyris element are depicted in the third figure in Figure 3, where $\bigcirc$ and $\uparrow$ (here and in the following examples) represent pointwise evaluation of second order derivatives and the normal derivative of the shape functions, respectively.

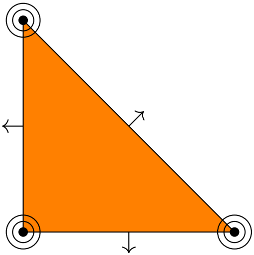


Figure: Quintic Argyris

Proof.

Suppose $u \in \mathbb{P}_5(T)$ with all dofs zero. Along an edge, u is a quintic polynomial with 2 treble roots, so $u = 0$ along each edge. Moreover, $\nabla u \cdot n$ is a quartic polynomial with 2 double roots and a single root, hence zero. Thus, u is divisible by $L_1^2 L_2^2 L_3^2$, which is of degree 6 . Thus $u = 0$.

Remark

The Argyris element with polynomials of degree five and 21 degrees of freedom within a single triangle, is the lowest order C^1 element.

Bell element.

By removing the nodal variables at the midpoints of the edges in the Argyris element and reducing the space of shape functions to

$$\{v \in \mathbb{P}_5(T) : (\partial v/\partial n)|_e \in \mathbb{P}_3(e) \text{ for each edge } e\},$$

we obtain the Bell element.

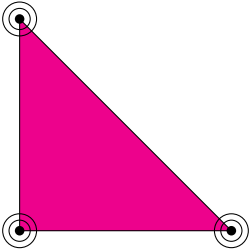


Figure: Bell

the Bell element represents a C^1 element, it has fewer degrees of freedom than the Argyris element.

Creuzeix-Raviart element

Let K be a triangle in $\mathbb{R}^2$ with vertices $\{\boldsymbol{z}_i\}_{i\in\{1,2,3\}}$. Let E_i be the face of T opposite to $\boldsymbol{z}_i$.

Creuzeix-Raviart element

Let K be a triangle in $\mathbb{R}^2$ with vertices $\{\boldsymbol{z}_i\}_{i\in\{1,2,3\}}$. Let E_i be the face of T opposite to $\boldsymbol{z}_i$.

The Crouzeix-Raviart finite element is defined by setting:

- The shape functions: $\mathcal{P} := \mathbb{P}_1$
- The degrees of freedom N_i

$$N_i(p) := \frac{1}{|E_i|}\int_{E_i} p \,\mathrm{d}s, \quad \forall i \in \{1,2,3\}$$

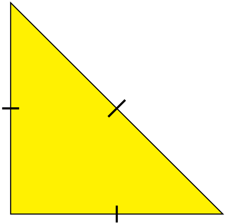


Figure: CR-element

The local interpolation operator :

$$\mathcal{I}_K^{\mathrm{CR}} : V(K) := W^{1,1}(K) \to P_1(K)$$

is such that

$$\mathcal{I}_K^{\mathrm{CR}}(v) := \sum_{i \in \{1,2,3\}} N_i(v)\phi_{K,i} \quad \text{for all } v \in V(K),$$

where $\{\phi_{K,i}\}_{i\in\{1,2,3\}}$ are the local shape functions in K s.t. $N_i(\phi_{K,j}) = \delta_{ij}$ for all $i, j \in \{1,2,3\}$. Recall that:

$$\phi_{K,i} := 1 - 2\lambda_i,$$

where $\{\lambda_i\}_{i\in\{1,2,3\}}$ are the barycentric coordinates in K.

Lemma (Local interpolation)

There is c s.t. for all $r \in [0,1]$, all $p \in [1,\infty]$, all $v \in W^{1+r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,

$$\left\| v - \mathcal{I}_K^{\mathrm{CR}}(v) \right\|_{L^p(K)} + h_K \left| v - \mathcal{I}_K^{\mathrm{CR}}(v) \right|_{W^{1,p}(K)} \le c h_K^{1+r} |v|_{W^{1+r,p}(K)} \quad (2)$$

Morley element

Morley element is $(T, \mathbb{P}_2, \mathcal{N})$, where the set $\mathcal{N}$ consists of evaluations of the shape functions at the vertices of T and the evaluations of the normal derivatives of the shape functions at the midpoints of the edges of T

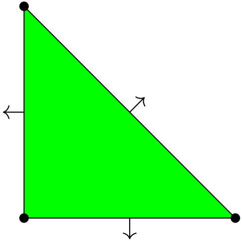


Figure: Morley

The continuity of the Morley element is very weak (non-C^0 continuity element). Morley element is the simplest nonconforming element for fourth order problems. It only consists of piecewise quadratic functions on every triangle

Morley interpolant

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size h. The Morley finite element space associated with $\mathcal{T}_h$ (cf. [20]) is defined by

$V_h = \{v \in L_2(\Omega) : v|_T \in \mathbb{P}_2(T), v$ is continuous at the vertices and the nor

of v is continuous at the midpoints of the edges $\}$,
Let $\Pi_h : H^2(\Omega) \longrightarrow V_h$ be the interpolation operator defined by the following conditions:

$$(\Pi_h\zeta)(p) = \zeta(p) \quad \forall p \in \mathcal{V}_h$$
$$\int_e \frac{\partial(\Pi_h\zeta)}{\partial n} ds = \int_e \frac{\partial\zeta}{\partial n} ds \quad \forall e \in \mathcal{E}_h$$

where ε_h is the set of all the edges of $\mathcal{T}_h$. Then we have:

$$\|\zeta - \Pi_h\zeta\|_h \leq Ch^\alpha |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H^{2+\alpha}(\Omega)$$

where $\|\cdot\|_h = \sqrt{a_h(\cdot,\cdot)}$ is the energy norm.

The Zienkiewicz element.

By removing the nodal variable at the center in the cubic Hermite element and reducing the space of shape functions to

$$\mathbb{P}_{3,red}(T) =: \left\{ v \in \mathbb{P}_3(T) : 6v(c) - \sum_{i=1}^{3} [2v(p_i) + (\nabla v)(p_i) \cdot (p_i - c)] = 0 \right\}$$

where $p_i (i = 1, 2, 3)$ and c are the vertices and center of T respectively, we obtain the Zienkiewicz element.

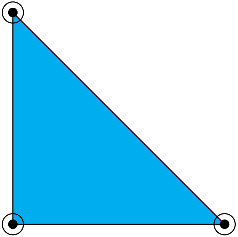


Figure: Zienckiez element

Raviart-Thomas element

In the finite element community, the symbol $\Diamond$ means the integral of the normal component, i.e.,

$$N_i(v) = \int_{e_i} v \cdot n_{e_i}\, ds, \quad i = 1, 2, 3$$

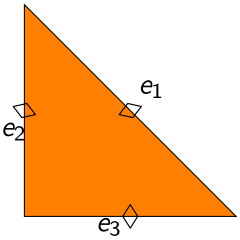


Figure: RT0

The local space is $RT_k(T) = [\mathbb{P}_k(T)]^2 \oplus \mathbf{x}\,\mathbb{P}_k(T)$, and its dimension is $\dim RT_k = (k+1)(k+3)$. The degrees of freedom (DOFs) are:

- Edge DOFs (normal moments). On each edge e

$$\boxed{\int_e (\mathbf{v}\cdot\mathbf{n})\, q\, ds \qquad \forall q \in \mathbb{P}_k(e)}$$

  Since $\dim \mathbb{P}_k(e) = k+1 \implies$ edge DOFs $= 3(k+1)$.

- Inside the element,

$$\boxed{\int_T \mathbf{v}\cdot\mathbf{q}\, dx \qquad \forall\, \mathbf{q} \in [\mathbb{P}_{k-1}(T)]^2}$$

  Since

$$\dim \mathbb{P}_{k-1}(T) = \frac{k(k+1)}{2} \implies \text{interior DOFs} = 2\cdot\frac{k(k+1)}{2} = k(k+1).$$

Arnold-Winther element

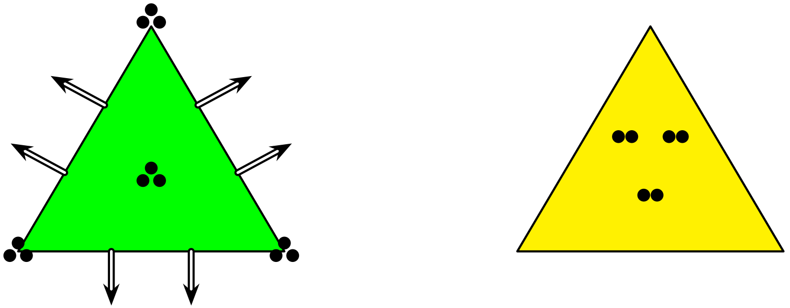


Figure: Element diagrams for the lowest order stress and displacement elements.

Arnold-Winther element

First we describe the finite elements on a single triangle $T \subset \Omega$. Define

$$\begin{aligned}\Sigma_T &= \mathcal{P}_2(T, \mathbb{S}) + \{\tau \in \mathcal{P}_3(T, \mathbb{S}) \mid \operatorname{div} \tau = 0\} \\ &= \{\tau \in \mathcal{P}_3(T, \mathbb{S}) \mid \operatorname{div} \tau \in \mathcal{P}_1(T, \mathbb{R}^2)\}, \\ V_T &= \mathcal{P}_1(T, \mathbb{R}^2).\end{aligned} \tag{3}$$

Here, $\mathbb{S}$ is the space of 2×2 symetric matrices. The space V_T has dimension 6 and a complete set of degrees of freedom are given by the values of the two components at the three nodes interior to T. The space Σ_T clearly has dimension at least 24, since

$$\dim \mathcal{P}_3(T, \mathbb{S}) = 30$$

and the condition that $\operatorname{div} \tau \in \mathcal{P}_1(T, \mathbb{R}^2)$ represents six linear constraints.

Arnold-Winther element

We now exhibit 24 degrees of freedom $\Sigma_T \to \mathbb{R}$ and show that they vanish simultaneously only when $\tau = 0$. This implies that the dimension of Σ_T is precisely 24, which could also be established directly using the fact that

$$\operatorname{div} \mathcal{P}_3(\Omega, \mathbb{S}) = \mathcal{P}_2(\Omega, \mathbb{R}^2)$$

and that the degrees of freedom are unisolvent. The degrees of freedom are

- the values of the three components of $\tau(x)$ at each vertex x of T (9 degrees of freedom),
- the values of the moments of degree 0 and 1 of the two normal components of τ on each edge e of T (12 degrees of freedom),
- the value of the three components of the moment of degree 0 of τ on T (3 degrees of freedom).

1. Vertex degrees of freedom (9 DOFs).
At each vertex x_i, $i = 1, 2, 3$, we prescribe the three independent components of τ:

$$\boxed{\tau_{11}(x_i), \qquad \tau_{12}(x_i), \qquad \tau_{22}(x_i).}$$

This gives $3 \times 3 = 9$ degrees of freedom.

2. Edge degrees of freedom (12 DOFs).
On each edge e we consider the traction vector

$$\tau n_e.$$

Its components in the normal–tangential frame are

$$(\tau n_e) \cdot n_e, \qquad (\tau n_e) \cdot t_e.$$

We define moments of degree 0 and 1 of these components.

Degree 0 *moments:*

$$\boxed{\int_e (\tau n_e)\cdot n_e\, ds, \qquad \int_e (\tau n_e)\cdot t_e\, ds.}$$

Degree 1 *moments:*

$$\boxed{\int_e s\,(\tau n_e)\cdot n_e\, ds, \qquad \int_e s\,(\tau n_e)\cdot t_e\, ds.}$$

Each edge contributes 4 degrees of freedom, hence

$$3 \times 4 = 12 \text{ edge DOFs.}$$

3. Element (cell) degrees of freedom (3 DOFs).

These are the degree 0 moments of τ over T:

$$\boxed{\int_T \tau_{11}\, dx, \qquad \int_T \tau_{12}\, dx, \qquad \int_T \tau_{22}\, dx.}$$

Total number of DOFs.

$$9 + 12 + 3 = 24.$$

These degrees of freedom are unisolvent for the space

$$\Sigma_T = \{\tau \in \mathcal{P}_3(T, \mathbb{S}) \mid \operatorname{div} \tau \in \mathcal{P}_1(T, \mathbb{R}^2)\}.$$

Finite Element Tables

Finite Element Method Tables (Arnold et al.)
https://www-users.cse.umn.edu/~arnold/femtable/index.html