Lecture 3: Finite element approximation of variational inequalities

Pr. Ismail Merabet

Univ. of K-M-Ouargla

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Contents

Introduction

In the first semester, we considered the finite element approximation for **variational equations**, i.e., problems of the form:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,v) = (f,v), \forall v \in V. \end{cases} \tag{1}$$

When the bilinear form a is **symmetric**, this problem can also be formulated as a **minimization** problem. Indeed, we define the functional

$$J(v) = \frac{1}{2} a(v,v) - (f,v)\,. \tag{2}$$

Then, problem (1) is equivalent to the following minimization problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ J(u) = \min_{v \in V} J(v) \end{cases} \tag{3}$$

In this first section of the second semester, we study minimization problems of the form (3) **but**, the solution u is sought in a **closed convex** set $\mathcal{K}$ instead of the Hilbert vector space V.

Thus, we are interested in problems of the form:

$$\begin{cases} \text{Find } u \in \mathcal{K} \text{ such that} \\ \quad J(u) = \min\limits_{v \in \mathcal{K}} J(v) \end{cases} \tag{4}$$

Theorem

If u is a solution of problem (4), then

$$a(u, v-u) \geq (f, v-u), \forall v \in \mathcal{K}.$$

Proof.

Let u be the solution of the minimization problem and $v \in \mathcal{K}$. Then,

$$J(u) \leq J((1-t)u + tv) \iff J(u) \leq J(u + t(v-u))$$

Expanding the terms, we obtain for all $v \in \mathcal{K}$

$$\begin{aligned} \frac{1}{2}a(u,u) - (f,u) = J(u) &\leq J(u + t(v-u)) \\ &= \frac{1}{2}a(u + t(v-u), u + t(v-u)) - (f, u + t(v-u)) \end{aligned}$$

Proof

Since the inequality holds for all $0 < t < 1$, we can divide by t to obtain

$$0 \leq a(u, v-u) - (f, v-u) + \frac{t}{2} a(v-u, v-u).$$

Taking the limit as $t \to 0$, we find that u must satisfy

$$a(u, v-u) \geq (f, v-u), \forall v \in \mathcal{K}.$$

Theorem (Stampacchia)

Let V be a Hilbert space and $\mathcal{K}$ a non-empty closed convex subset of V. Suppose that $a(\cdot,\cdot)$ is a continuous and coercive bilinear form on V. Given $f \in V'$, there exists a unique $u \in \mathcal{K}$ such that

$$a(u, v - u) \geq \langle f, v - u \rangle, \quad \forall v \in \mathcal{K}. \tag{5}$$

Proof.

By the Riesz-Frechét representation theorem, there exists a unique $\varphi \in V$ such that:

$$\langle f, v \rangle = (\varphi, v)_V, \quad \forall v \in V.$$

On the other hand, for any fixed $u \in V$, the mapping $v \mapsto a(u, v)$ is a continuous linear form on V. By the Riesz-Frechét representation theorem, there exists an element $Au \in V$ such that

$$a(u, v) = (Au, v), \quad \forall v \in V.$$

It is clear that A is a linear operator from V to V and satisfies

$$\|Aw\| \leq C\|w\|, \quad \forall w \in V, \tag{6}$$

$$(Aw, w) \geq \alpha\|w\|^2, \forall w \in V. \tag{7}$$

The problem thus reduces to finding $u \in \mathcal{K}$ such that

$$(Au, v - u) \geq (\varphi, v - u), \quad \forall v \in \mathcal{K}. \tag{8}$$

For some constant $\rho > 0$ to be determined, the inequality (8) is equivalent to

$$(\rho\varphi - \rho Au + u - u, v - u) \leq 0, \tag{9}$$

$$u = P_{\mathcal{K}}(\rho\varphi - \rho Au + u).$$

For any $v \in \mathcal{K}$, define $Tv = P_{\mathcal{K}}(\rho\varphi - \rho Av + v)$. We show that if $\rho > 0$ is suitably chosen, then T is a strict contraction, i.e.,

$$\|Tv_1 - Tv_2\| \leq k\|v_1 - v_2\|, \forall v_1, v_2 \in \mathcal{K}.$$

Since $P_{\mathcal{K}}$ is a projection, we have

$$\|Tv_1 - Tv_2\| \leq \|(v_1 - v_2) - \rho(Av_1 - Av_2)\|$$

and thus

$$\begin{aligned}\|Tv_1 - Tv_2\|^2 &\leq \|(v_1 - v_2)\|^2 - 2\rho\left((Av_1 - Av_2), v_1 - v_2\right) + \rho^2\|Av_1 - Av_2\|^2 \\ &\leq \|v_1 - v_2\|^2(1 - 2\rho\alpha + \rho^2 C^2).\end{aligned}$$

Choosing $\rho > 0$ such that

$$k^2 = 1 - 2\rho\alpha + \rho^2 C^2 < 1 \qquad (0 < \rho < \frac{2\alpha}{C^2}),$$

then for this choice, T has a unique fixed point.

Let Ω be a bounded open subset of $\mathbb{R}^2$ with boundary $\partial\Omega$. Given $f \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega) \cap C(\bar{\Omega})$, we define:

$$
\begin{aligned}
V :=& H_0^1(\Omega) := \{v \in H^1(\Omega) \quad | \quad v = 0 \text{ on } \partial\Omega\}, \\
\mathcal{K} :=& \{v \in V; \quad v \geq \psi \text{ a.e. in } \Omega\}.
\end{aligned}
$$

We are interested in analyzing the following variational inequality:

$$
\begin{cases}
\text{Find } u \in \mathcal{K} \text{ such that} \\
\displaystyle\int_\Omega \nabla u \cdot \nabla(v - u) dx \geq \int_\Omega f(v - u) dx, \quad \forall v \in \mathcal{K}.
\end{cases}
\tag{10}
$$

The search for the vertical displacement consists of finding u in the set:

$$
\mathcal{K} = \{v \in H_0^1(\Omega) \mid \quad v \geq \psi\}
$$

Proposition

The set $\mathcal{K}$ is convex and closed in $H_0^1(\Omega)$.

Proof

To prove that $\mathcal{K}$ is convex, it is necessary and sufficient to show that:

$$v_1, v_2 \in \mathcal{K} \implies tv_1 + (1-t)v_2 \geq \psi, \quad \forall t \in (0,1)$$

Indeed, let $t \in (0,1)$, then we have $tv_1 \geq t\psi$ and $(1-t)v_2 \geq (1-t)\psi$, hence:

$$tv_1 + (1-t)v_2 \geq (t+1-t)\psi = \psi$$

Let $v_n \in \mathcal{K}$ such that $\|v - v_n\|_{H^1(\Omega)} \to 0$ as $n \to \infty$.

$$\psi(x) \leq v_n(x) - v(x) + v(x) \leq \|v_n - v\| + v(x) \implies v(x) \geq \psi$$

Proposition

The form $a(\cdot,\cdot)$ defined by $a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$ is bilinear, continuous, and coercive on $H_0^1(\Omega)$.

Theorem

The solution u of (10) satisfies the complementarity system:

$$\begin{cases} -\Delta u - f \geq 0, & \text{in } \Omega \\ u - \psi \geq 0, & \text{in } \Omega \\ (u - \psi, -\Delta u - f) = 0, & \text{in } \Omega \end{cases} \tag{11}$$

Proof

Let $\varphi \in C_0^\infty(\Omega), \varphi \geq 0$, then $v = u + \varphi \in \mathcal{K}$, and (10) implies

$$\int_\Omega \nabla u \cdot \nabla \varphi dx \geq \int_\Omega f \varphi dx$$

By integration by parts, we obtain:

$$-\int_\Omega \Delta u \; \varphi \geq \int_\Omega f \varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \varphi \geq 0$$

which leads to $-\Delta u - f \geq 0, \quad \text{in } \Omega$.

The second inequality and the fourth equation follow from the definition of $\mathcal{K}$.

It remains to prove the third equation.

Indeed, we decompose $\Omega = \mathcal{O} \cup \mathcal{C}$ with $\mathcal{C}$ being the contact set, i.e.,

$$\mathcal{C} = \{x \in \Omega \quad | \quad u = \psi\} \quad \text{and} \quad \mathcal{O} = \{x \in \Omega \quad | \quad u > \psi\}$$

It is not difficult to see that $\mathcal{O}$ is open and $\mathcal{C}$ is closed.

It is clear that if $x \in \mathcal{C}$, then $u - \psi = 0$ and thus

$$(u - \psi)(-\Delta u - f) = 0 \quad \text{in } \mathcal{C}.$$

Now, if $x \in \mathcal{O}$, which is an open set, we take $v = u \pm \varepsilon\varphi$, $\forall \varphi \in C_0^\infty(\mathcal{O}), \varphi \geq 0$ and integrate by parts over $\mathcal{O}$, then we obtain

$$-\Delta u = f \quad \text{in } \mathcal{O}$$

and thus

$$(u - \psi)(-\Delta u - f) = 0 \quad \text{in } \Omega$$

Conversely, if u is a solution of the complementarity system, then u is the solution of the variational inequality. Indeed, for all $v \in \mathcal{K}$, we have,

$$\begin{cases} (-\Delta u - f)(v - \psi) \geq 0 \\ (-\Delta u - f)(u - \psi) = 0 \end{cases} \Longrightarrow (-\Delta u - f, v - \psi - u + \psi) \geq 0 \tag{12}$$

which leads to

$$\int_\Omega \nabla u \cdot \nabla (v - u) \geq (f, v - u), \quad \forall v \in \mathcal{K}.$$

Theorem

If $\partial\Omega$ is sufficiently regular ($C^{1,1}$), then $u \in H^2(\Omega) \cap H_0^1(\Omega)$.

Theorem (Stampacchia-Brezis)

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular open set ($C^{1,1}$), $f \in L^2(\Omega)$, and $\psi \in H^2(\Omega)$. Then the solution u of (10) belongs to $H^2(\Omega)$.

The corresponding minimization problem is:

$$u = \arg\min_{\mathcal{K}} J \quad \text{with} \quad J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \langle f, v\rangle$$

$$\mathcal{K} = \{v \in H_0^1(\Omega) \quad | \quad v \geq \psi\}$$

The formulation as a variational inequality takes the form:

$$\begin{cases} \text{Find } u \in \mathcal{K} = \{v \in H_0^1(\Omega) \quad | \quad v \geq \psi \text{ in } \Omega\} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_\Omega f(v-u)dx, \quad \forall v \in \mathcal{K} \end{cases} \tag{13}$$

Define $\tilde{u} = u - \psi$, then

$$\int_\Omega \nabla u \cdot \nabla(v - u)dx \geq \int_\Omega f(v - u)dx, \quad \forall v \in \mathcal{K}$$

implies that $\forall v \in \mathcal{K}$:

$$\begin{aligned}\int_\Omega \nabla(u - \psi) \cdot \nabla(v - \psi - (u - \psi))dx &\geq \int_\Omega f(v - \psi - (u - \psi))dx - \int_\Omega \nabla\psi \cdot \\ &= \langle f + \Delta\psi, v - \psi - (u - \psi)\rangle\end{aligned}$$

thus

$$\int_\Omega \nabla\tilde{u} \cdot \nabla(\tilde{v} - \tilde{u})dx \geq \int_\Omega (f + \Delta\psi)(\tilde{v} - \tilde{u})dx, \quad \forall\tilde{v} \in \tilde{\mathcal{K}}$$

$$\tilde{u} = \arg\min_{\tilde{K}} \tilde{J} \quad \text{with} \quad \tilde{J}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \left\langle \tilde{f}, v \right\rangle$$

$$\tilde{\mathcal{K}} = \{v \in H_0^1(\Omega) \quad | \quad v \geq 0\}, \quad \text{and} \quad \tilde{f} = f + \Delta\psi$$

We have already seen (see the complementarity system)

$$\Delta\tilde{u} \leq -\tilde{f} \quad \text{a.e. in } \Omega$$

Let $v \in H_0^1(\Omega)$, $v \geq 0$ and define $v_\varepsilon(x) = v(x)\varphi(\frac{\tilde{u}(x)}{\varepsilon})$ where

$$\varphi(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x \geq 2. \end{cases} \tag{14}$$

Since we have

$$v_\varepsilon(x) \neq 0 \implies \tilde{u}(x) > \varepsilon$$

Then

$$v_\varepsilon \geq 0 \quad \text{and} \quad \text{supp } v_\varepsilon \subset \mathcal{O} = \{x \in \Omega \quad | \quad \tilde{u}(x) > 0\}$$

$$\int \nabla \tilde{u} \cdot \nabla v_\varepsilon + \int \tilde{f} v_\varepsilon = 0 \tag{15}$$

Using the chain rule for differentiation, we find

$$\nabla v_\varepsilon = \nabla v \varphi(\tilde{u}/\varepsilon) + v\varphi'(\tilde{u}/\varepsilon)\frac{1}{\varepsilon}\nabla \tilde{u}$$

thus

$$\int \nabla\tilde{u}\cdot(\nabla v\varphi(\tilde{u}/\varepsilon) + v\varphi'(\tilde{u}/\varepsilon)\frac{1}{\varepsilon}\nabla\tilde{u}) + \int \tilde{f}v\varphi(\tilde{u}/\varepsilon)$$
$$= \int \nabla\tilde{u}\cdot\nabla v\varphi(\tilde{u}/\varepsilon) + \underbrace{\frac{1}{\varepsilon}\int v\varphi'(\tilde{u}/\varepsilon)|\nabla\tilde{u}|^2}_{\geq 0} + \int \tilde{f}v\varphi(\tilde{u}/\varepsilon)$$
$$= 0$$

which implies

$$\int \nabla\tilde{u}\cdot\nabla v\varphi(\tilde{u}/\varepsilon) + \int \tilde{f}v\varphi(\tilde{u}/\varepsilon) \leq 0$$

Taking the limit as $\varepsilon \to 0$ gives

$$\int \nabla\tilde{u}\cdot\nabla v\chi_{\{\tilde{u}>0\}} + \int \tilde{f}v\chi_{\{\tilde{u}>0\}} \leq 0$$

which means

$$\Delta\tilde{u} \geq \tilde{f}\chi_{\{\tilde{u}>0\}}$$

Finally, we obtain $\tilde{f}\chi_{\{\tilde{u}>0\}} \leq \Delta\tilde{u} \leq -\tilde{f} \quad \Longrightarrow \Delta\tilde{u} \in L^2(\Omega)$. By the elliptic regularity theorem, we have $\tilde{u} \in H^2(\Omega)$.

Let V_h be a finite-dimensional space $V_h \subset V$, and we construct a convex and closed subset $\mathcal{K}_h \subset V_h$:

$$\mathcal{K}_h = \{v_h \in V_h, \quad v_h \geq \psi_h\}$$

where ψ_h is the interpolant of ψ in V_h. We consider the following problem:

$$\begin{cases} \text{Find } u_h \in \mathcal{K}_h \text{ such that} \\ \displaystyle\int_\Omega \nabla u_h \cdot \nabla (v_h - u_h) dx \geq \int_\Omega f(v_h - u_h) dx, \quad \forall v_h \in \mathcal{K}_h \end{cases} \tag{16}$$

Remark: The major difficulty in error analysis for variational inequalities lies in the fact that even if $V_h \subset V$, the set $\mathcal{K}_h$ is not necessarily included in $\mathcal{K}$. Indeed, if $\psi_h \leq \psi$, then $v_h \geq \psi_h$ does not necessarily imply that $v_h \geq \psi$.

Theorem (Brezzi-Hager-Raviart)

Let $u \in H^2(\Omega) \cap \mathcal{K}$ be the solution of (10) and $u_h \in \mathcal{K}_h$ (with $V_h = V_h^1$) be the solution of (16). Then, we have:

$$|u - u_h|_{H^1(\Omega)} = \|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C\ h\left(|u|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)} + |\psi|_{H^2(\Omega)}\right)$$

$$\begin{aligned}\|\nabla(u - u_h)\|^2_{L^2(\Omega)} &= a(u - u_h, u - u_h) \\ &= \underbrace{a(u - u_h, u - \mathcal{I}_h u)}_{\mathrm{I}} + \underbrace{a(u - u_h, \mathcal{I}_h u - u_h)}_{\mathrm{II}}\end{aligned}$$

For term I, using the Cauchy-Schwarz inequality and $ab \leq \dfrac{a^2}{2} + \dfrac{b^2}{2}$, we get:

$$\begin{aligned}a(u - u_h, u - \mathcal{I}_h u) &\leq \frac{\|\nabla(u - u_h)\|^2_{L^2(\Omega)}}{2} + \frac{\|\nabla(u - \mathcal{I}_h u)\|^2_{L^2(\Omega)}}{2} \\ &\leq \frac{\|\nabla(u - u_h)\|^2_{L^2(\Omega)}}{2} + C\ h^2|u|^2_{H^2(\Omega)}\end{aligned}$$

For term II, we have:

$$\begin{aligned}
a(u-u_h,\mathcal{I}_h u-u_h) =& a(u,\mathcal{I}_h u-u_h)-a(u_h,\mathcal{I}_h u-u_h)\\
\leq& a(u,\mathcal{I}_h u-u_h)-(f,\mathcal{I}_h u-u_h)\\
=& (\nabla u,\nabla(\mathcal{I}_h u-u_h))-(f,\mathcal{I}_h u-u_h)\\
=& (-\Delta u-f,\mathcal{I}_h u-u_h)\\
=& (\lambda,\mathcal{I}_h u-u_h)
\end{aligned}$$

By further estimations, we obtain:

$$\begin{aligned}
\frac{\|\nabla(u-u_h)\|^2_{L^2(\Omega)}}{2} &\leq C\left(\; h^2|u|^2_{H^2(\Omega)}+h\|\lambda\|_{L^2(\Omega)}\; h|u-\psi|_{H^2(\Omega)}\right)\\
&\leq C\left(\; h^2|u|^2_{H^2(\Omega)}+\frac{h^2\|\lambda\|^2_{L^2(\Omega)}}{2}\;+\frac{h^2|u-\psi|^2_{H^2(\Omega)}}{2}\right)
\end{aligned}$$