Lecture 4: Finite element method for parabolic equations I

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Contents

Introduction

The purpose is to present a second important extension of the linear elliptic Poisson problem, this time into a time-dependent partial differential equation. We investigate :

- ☞ the existence and uniqueness of a weak solution.
- ☞ energy estimates.
- ☞ appropriate finite element discretization.
- ☞ stability of the discrete problem.
- ☞ error estimates.

We will follow the book of Ern and Guermond:

- Ern, A., and Guermond, J-L. Finite Elements III. First-Order and Time-Dependent PDEs

The heat equation reads as follows:
for a final time $T > 0$ and source term $f \in L^2\left(0, T; L^2(\Omega)\right)$, find a scalar-valued function $u : \Omega \times (0, T) \to \mathbb{R}$ such that

$$
\begin{aligned}
\partial_t u - \Delta u &= f \qquad && \text{in } \Omega \times (0, T), \\
u &= 0 && \text{on } \partial\Omega \times (0, T), \\
u(0) &= 0 && \text{in } \Omega.
\end{aligned}
\tag{1}
$$

Bochner function spaces are a generalization of Lebesgue spaces to functions whose values lie in a Banach space, instead of real or complex numbers, cf. Ern and Guermond [Section 56.1]. We will in particular need the function space with weak partial derivatives with respect to the spatial variables to belong to L^2 in both space and time

$$X := L^2\left(0, T; H_0^1(\Omega)\right). \tag{2}$$

We will also need its subspace additionally requesting the weak partial derivative with respect to the time variable to belong to H^{-1} in space and L^2 in time,

$$Y := \left\{v \in X; \quad \partial_t v \in L^2\left(0, T; H^{-1}(\Omega)\right)\right\} \tag{3}$$

We note that

$$Y = L^2\left(0, T; H_0^1(\Omega)\right) \cap H^1\left(0, T; H^{-1}(\Omega)\right) \subset C\left(0, T; L^2(\Omega)\right).$$

We will also impose the zero initial condition in the subspace

$$Y_0 := \{v \in Y; \quad v(0) = 0\}. \tag{4}$$

We equip the spaces X and Y with the following norms:

$$\|v\|_X^2 := \int_0^T \|\nabla v\|^2 \,\mathrm{d}t, \quad v \in X, \tag{5}$$

$$\|v\|_Y^2 := \int_0^T \|\nabla v\|^2 + \|\partial_t v\|_{H^{-1}(\Omega)}^2 \,\mathrm{d}t + \|v(T)\|^2, \quad v \in Y_0, \tag{6}$$

where $H^{-1}(\Omega)$ is the dual space to $H_0^1(\Omega)$. With $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$,

$$\|v\|_{H^{-1}(\Omega)} = \sup_{\substack{\varphi \in H_0^1(\Omega) \\ \|\nabla \varphi\| = 1}} \langle v, \varphi \rangle. \tag{7}$$

The weak formulation for the heat equation is:
Find $u \in Y_0$ such that

$$\int_0^T \langle \partial_t u, v \rangle + (\nabla u, \nabla v) \mathrm{d}t = \int_0^T (f, v) \mathrm{d}t \quad \forall v \in X. \tag{8}$$

One observs that in contrast to the developments of the previous chapters, one looks for the weak solution in the trial space Y, which is different from the test space X. This is in line with the nonsymmetry between space and time in (1). This is also the origin of the fact that the analysis of (8) will be more involved. We also remark that formulation (8) is equivalent to finding $u \in Y_0$ such that

$$\langle \partial_t u(t), v \rangle + (\nabla u(t), \nabla v) = (f(t), v) \quad \forall v \in H_0^1(\Omega), \text{ for a.e. } t \in (0, T). \tag{9}$$

Recall that

$$\|\nabla\varphi\| = \max_{\substack{v\in H_0^1(\Omega)\\ \|\nabla v\|=1}} (\nabla\varphi, \nabla v) = \max_{v\in H_0^1(\Omega)} \frac{(\nabla\varphi, \nabla v)}{\|\nabla v\|} \quad \forall\varphi \in H_0^1(\Omega), \tag{10}$$

Note that:

$$\forall\varphi \in X, \quad \|\varphi\|_X = \max_{v\in X\setminus\{0\}} \frac{\int_0^T (\nabla\varphi, \nabla v)\mathrm{d}t}{\|v\|_X}. \tag{11}$$

Indeed, on the one hand,

$$\max_{v\in X} \frac{\int_0^T (\nabla\varphi, \nabla v)\mathrm{d}t}{\|v\|_X} \le \|\varphi\|_X \max_{v\in X} \frac{\|v\|_X}{\|v\|_X} = \|\varphi\|_X$$

by virtue of (5) and the Cauchy-Schwarz inequality. On the other hand,

$$\|\varphi\|_X = \frac{\|\varphi\|_X^2}{\|\varphi\|_X} = \frac{\int_0^T (\nabla\varphi, \nabla\varphi)\mathrm{d}t}{\|\varphi\|_X} \le \max_{v\in X} \frac{\int_0^T (\nabla\varphi, \nabla v)\mathrm{d}t}{\|v\|_X}$$

where the lower bound follows by picking $\varphi \in X$ in the max.

The following is a central result for problem (1)

Theorem (Inf-sup identity)

For every $\varphi \in Y_0$, there holds

$$\|\varphi\|_Y = \max_{v \in X} \frac{\int_0^T \langle \partial_t \varphi, v\rangle + (\nabla\varphi, \nabla v)\mathrm{d}t}{\|v\|_X} \tag{12}$$

Proof

For a fixed $\varphi \in Y_0$, let $w_* \in X$ be defined by, a.e. in $(0, T)$,

$$(\nabla w_*, \nabla v) = \langle \partial_t \varphi, v\rangle \quad \forall v \in H_0^1(\Omega). \tag{13}$$

Using (10) and (7), this implies the identity

$$\|\nabla w_*\| = \|\partial_t \varphi\|_{H^{-1}(\Omega)}, \text{ a.e. in } (0, T), \tag{14}$$

as well as

$$\int_0^T \langle \partial_t \varphi, v\rangle + (\nabla\varphi, \nabla v)\mathrm{d}t = \int_0^T (\nabla(w_* + \varphi), \nabla v)\,\mathrm{d}t \quad \forall v \in X.$$

Consequently, using (11),

$$\|w_* + \varphi\|_X = \max_{v \in X} \frac{\int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) \mathrm{d}t}{\|v\|_X}. \tag{15}$$

Moreover, the following useful identity holds true on the space Y_0 :

$$2 \int_0^T \langle \partial_t \varphi, \varphi \rangle \, \mathrm{d}t = \int_0^T \frac{d}{dt} \|\varphi\|^2 \, \mathrm{d}t = \|\varphi(T)\|^2 - \|\varphi(0)\|^2 = \|\varphi(T)\|^2. \tag{16}$$

Consequently,

$$\begin{aligned} \|w_* + \varphi\|_X^2 &= \int_0^T \|\nabla (w_* + \varphi)\|^2 \, \mathrm{d}t \\ &= \int_0^T \|\nabla w_*\|^2 + 2 (\nabla w_*, \nabla \varphi) + \|\nabla \varphi\|^2 \, \mathrm{d}t \\ &= \int_0^T \|\partial_t \varphi\|_{H^{-1}(\Omega)}^2 + 2 \langle \partial_t \varphi, \varphi \rangle + \|\nabla \varphi\|^2 \, \mathrm{d}t = \|\varphi\|_Y^2, \end{aligned}$$

so that the claim (12) follows from (15).

Remark (Inf-sup condition).

One remarks easily that (12) in particular implies

$$\inf_{\varphi \in Y_0} \sup_{v \in X} \frac{\int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) \mathrm{d}t}{\|v\|_X \|\varphi\|_Y} \geq C. \tag{17}$$

We from (12) actually have an equality with $C = 1$ and min and max in place of inf and sup. The writing (17) gives rise to the nomenclature inf-sup condition, which is central in analysis of partial differential equations and finite element methods of nonsymmetric problems, In this subsection we prove the existence and the uniqueness of a weak solution by the Banach closed range and open mapping theorems. Let X' be the dual of X, thus,

$$X' = L^2\left(0, T; H^{-1}(\Omega)\right),$$

and let $\langle \cdot, \cdot \rangle_{X',X}$ denote the corresponding duality pairing.
Define the operator:

$$B_Y : Y_0 \to X'$$

by

$$\langle B_Y(\varphi), v \rangle_{X',X} := \int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) \mathrm{d}t \quad v \in X, \quad \varphi \in Y_0. \tag{18}$$

This operator is clearly linear and bounded as, for all $\varphi \in Y_0$,

$$\begin{aligned} \|B_Y(\varphi)\|_{X'} &= \sup_{v \in X} \frac{\langle B_Y(\varphi), v \rangle_{X',X}}{\|v\|_X} \\ &\overset{(18)}{=} \sup_{v \in X} \frac{\int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) \mathrm{d}t}{\|v\|_X} \\ &\overset{(12)}{=} \|\varphi\|_Y \end{aligned} \tag{19}$$

The weak formulation (8) can then be equivalently rewritten as:
find $u \in Y_0$ such that

$$B_Y(\varphi) = f \quad \text{in } X'. \tag{20}$$

Let Y_0' be the dual of Y_0 and $\langle \cdot, \cdot \rangle_{Y_0', Y_0}$ the corresponding duality pairing. We will also need below the adjoint operator defined by

$$\begin{aligned} B_Y^* &: X \to Y_0' \\ \langle B_Y^*(v), \varphi \rangle_{Y_0', Y_0} &:= \langle B_Y(\varphi), v \rangle_{X', X} \quad v \in X, \quad \varphi \in Y_0. \end{aligned} \tag{21}$$

We can now state and prove the result of existence and uniqueness of $u \in Y_0$:

Theorem

There exists a unique solution $u \in Y_0$.

Since (8) is equivalent to (20), the existence and uniqueness of a weak solution $u \in Y_0$ follows when the operator B_Y from (18) is bijective. By the Banach closed range and open mapping theorems, this is in turn equivalent to showing that

$$i)\ B_Y \text{ is injective}, \qquad ii)\ B_Y \text{ is surjective} \tag{22}$$

$$i)\ B_Y^* \text{ is surjective}, \qquad ii)\ B_Y^* \text{ is injective}, \tag{23}$$

$$i)\ B_Y \text{ is injective}, \qquad ii)\ \text{range of } B_Y \text{ is closed in } X', \qquad iii)\ B_Y^* \text{ is injective}, \tag{24}$$

We will prove the three properties in (24).

Proof

1. Let $B_Y(\varphi) = 0$ for some $\varphi \in Y_0$. Then, by (18),

$$\int_0^T \langle \partial_t \varphi, v \rangle + (\nabla \varphi, \nabla v) \mathrm{d}t = 0 \quad v \in X.$$

By virtue of Theorem (inf-sup identity), this implies $\varphi = 0$, i.e., injectivity (actually, B_Y is automatically injective, since it is an isometry).

2. Consider a sequence $\varphi_i \in Y_0$ such that $B_Y(\varphi_i)$ is a Cauchy sequence in X'. Thus, for any real $\epsilon > 0$, there exists $k > 0$ such that for all $m > 0$,

$$\|B_Y(\varphi_{k+m}) - B_Y(\varphi_k)\|_{X'} \leq \epsilon.$$

This, however, immediately implies that φ_i is a Cauchy sequence in Y_0, since

$$\|B_Y(\varphi_{k+m}) - B_Y(\varphi_k)\|_{X'} = \|B_Y(\varphi_{k+m} - \varphi_k)\|_{X'} \overset{(19)}{=} \|\varphi_{k+m} - \varphi_k\|_Y.$$

Taking its limit $\varphi \in Y_0$, we obtain that

$$\lim_{k\to\infty} \|B_Y(\varphi) - B_Y(\varphi_k)\|_{X'} \overset{(19)}{=} \lim_{k\to\infty} \|\varphi - \varphi_k\|_Y = 0,$$

so that the range of B_Y is closed in X'.

iii) Finally, let $B_Y^*(v) = 0$ for some $v \in X$. Then, by (21) and (18),

$$\langle B_Y^*(v), \varphi\rangle_{Y_0', Y_0} = \langle B_Y(\varphi), v\rangle_{X', X} = \int_0^T \langle \partial_t \varphi, v\rangle + (\nabla\varphi, \nabla v)\mathrm{d}t = 0\, \forall \varphi \in Y \tag{25}$$

Define $\xi \in L^2\left(0, T; H^{-1}(\Omega)\right)$ by, a.e. in $(0, T)$,

$$\langle \xi, w \rangle = (\nabla v, \nabla w) \quad \forall w \in H_0^1(\Omega). \tag{26}$$

Then (25) in particular implies

$$\int_0^T \langle \partial_t \varphi, v \rangle \, \mathrm{d}t = -\int_0^T \langle \xi, \varphi \rangle \mathrm{d}t \quad \forall \varphi \in \mathcal{D}((0, T) \times \Omega) \subset Y_0,$$

which is the meaning of

$$\partial_t v = \xi \tag{27}$$

and in particular shows that, actually, $v \in Y$.

Consider now an arbitrary function $w \in H_0^1(\Omega)$, so that when multiplied by the time variable t, $tw \in Y_0$. Taking tw as a test function φ in (25) and using the integration by parts in time formula, we see

$$
\begin{aligned}
0 &= \int_0^T \langle \partial_t(tw), v\rangle + \langle \xi, tw\rangle \mathrm{d}t \\
&= \int_0^T \langle \partial_t(tw), v\rangle + \langle \partial_t v, tw\rangle \,\mathrm{d}t \\
&= T(w, v(T)) - 0 \\
&= T(w, v(T)).
\end{aligned}
$$

Since $w \in H_0^1(\Omega)$ was arbitrary and since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, we infer that

$$
v(T) = 0. \tag{28}
$$

We finally use tv as a test function $\varphi \in Y_0$ in (25). This gives, similarly as in (16),

$$
\begin{aligned}
\int_0^T \langle \partial_t(tv), v\rangle \,\mathrm{d}t &= -\int_0^T \langle \xi, tv\rangle \mathrm{d}t \\
&= -\int_0^T \langle \partial_t v, tv\rangle \,\mathrm{d}t \\
&= -\int_0^T t\,\langle \partial_t v, v\rangle \,\mathrm{d}t \\
&= -\frac{1}{2}\int_0^T t\frac{d}{dt}\|v\|^2 \,\mathrm{d}t \\
&= -\frac{1}{2}\left[T\|v(T)\|^2 - 0\|v(0)\|^2 - \int_0^T \|v\|^2 \,\mathrm{d}t\right] \\
&= \frac{1}{2}\int_0^T \|v\|^2 \,\mathrm{d}t.
\end{aligned}
$$

Consequently, still taking tv as a test function $\varphi \in Y_0$ in (25), and using this result,

$$0 = \int_0^T \langle \partial_t(tv), v\rangle + (\nabla(tv), \nabla v)\mathrm{d}t = \frac{1}{2}\int_0^T \|v\|^2 \,\mathrm{d}t + \int_0^T t\|\nabla v\|^2 \,\mathrm{d}t$$

From here, we conclude $v = 0$, i.e., the injectivity of B_Y^*.

Remark

Note that the analysis carried out in the previous subsection can be generalized to the problem

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } \Omega \times (0, T), \\ u &= 0 && \text{on } \partial\Omega \times (0, T), \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{29}$$

where $u_0 \in L^2(\Omega)$.

Define

$$X := L^2(\Omega) \times L^2(0, T, H_0^1(\Omega)) \tag{30}$$

$$Y := L^2(0, T, H_0^1(\Omega)) \cap H^1(0, T, H^{-1}(\Omega)) \tag{31}$$

$$\mathcal{B}_{v_0}(w, v) := (w(0), v_0) + \int_0^T \langle \partial_t w, v \rangle + (\nabla w, \nabla v)\mathrm{d}t \tag{32}$$

$$\ell_{v_0}(v) := (u_0, v_0) + \int_0^T (f, v) dt \tag{33}$$

Then the weak form reads:

$$\begin{cases} \text{Find } u \in Y \quad \text{ such that :} \\ \quad \mathcal{B}_{v_0}(u, v) = \ell_{v_0}(v), \quad \forall (v_0, v) \in X \end{cases} \tag{34}$$