Lecture 5: Finite element method for parabolic equations II

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Contents

Energy estimates for the heat equation

Let us consider the following heat equation:

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } \Omega \times (0, T), \\ u &= 0 && \text{on } \partial\Omega \times (0, T), \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{1}$$

where $u_0 \in L^2(\Omega)$.

Theorem

There exists $C > 0$ such that:

$$\|u(\cdot, t)\|^2_{L_2(\Omega)} \leq \mathrm{e}^{-Ct} \|u_0\|^2_{L_2(\Omega)} + \frac{1}{C} \int_0^t \mathrm{e}^{-C(t-\tau)} \|f(\cdot, \tau)\|^2_{L_2(\Omega)} \, \mathrm{d}\tau. \tag{2}$$

proof

Taking the inner product of (1) with u, noting that $u(x,t)=0, x\in\partial\Omega$, integrating by parts, we get

$$\left(\frac{\partial u}{\partial t}(\cdot,t), u(\cdot,t)\right)+\sum_{i=1}^{n}\left\|\frac{\partial u}{\partial x_i}(\cdot,t)\right\|_{L_2(\Omega)}^2=(f(\cdot,t),u(\cdot,t))$$

Noting that

$$\left(\frac{\partial u}{\partial t}(\cdot,t), u(\cdot,t)\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|_{L_2(\Omega)}^2,$$

and using the Poincaré-Friedrichs inequality, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|_{L_2(\Omega)}^2+\frac{1}{c_p^2}\|u(\cdot,t)\|_{L_2(\Omega)}^2\le(f(\cdot,t),u(\cdot,t)).$$

Let $C = 1/c_p^2$; then, by the Cauchy-Schwarz inequality,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|^2_{L_2(\Omega)} + C\|u(\cdot,t)\|^2_{L_2(\Omega)} &\leq \|f(\cdot,t)\|_{L_2(\Omega)}\|u(\cdot,t)\|_{L_2(\Omega)} \\ &\leq \frac{1}{2C}\|f(\cdot,t)\|^2_{L_2(\Omega)} + \frac{C}{2}\|u(\cdot,t)\|^2_{L_2(\Omega)}.\end{aligned}$$

hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\cdot,t)\|^2_{L_2(\Omega)} + C\|u(\cdot,t)\|^2_{L_2(\Omega)} \leq \frac{1}{C}\|f(\cdot,t)\|^2_{L_2(\Omega)}.$$

Multiplying both sides by e^{Ct}

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{Ct}\|u(\cdot,t)\|^2_{L_2(\Omega)}\right) \leq \frac{\mathrm{e}^{Ct}}{K}\|f(\cdot,t)\|^2_{L_2(\Omega)}.$$

Integrating from 0 to t

$$\mathrm{e}^{Ct}\|u(\cdot,t)\|^2_{L_2(\Omega)} - \|u_0\|^2_{L_2(\Omega)} \leq \frac{1}{C}\int_0^t \mathrm{e}^{C\tau}\|f(\cdot,\tau)\|^2_{L_2(\Omega)}\,\mathrm{d}\tau$$

Remark

Estimates of the form (2) can be used to prove uniqueness of solution. Indeed, if u_1 and u_2 are solutions to (1), then $u = u_1 - u_2$ satisfies (2) with $f \equiv 0$ and $u_0 \equiv 0$; therefore, by (2), $u \equiv 0$, i.e. $u_1 \equiv u_2$.

Remark

Let us also look at the special case when $f \equiv 0$ in (1). This corresponds to considering the evolution of the solution from the initial datum u_0 in the absence of external forces. In this case (2) yields

$$\|u(\cdot,t)\|^2_{L_2(\Omega)} \le \mathrm{e}^{-Ct} \|u_0\|^2_{L_2(\Omega)}, \quad t \ge 0; \tag{4}$$

in physical terms, the energy $\frac{1}{2}\|u(\cdot,t)\|^2_{L_2(\Omega)}$ dissipates exponentially.

Semi-discretization in space

We consider the following semi-discrete trial and test spaces:

$$
\begin{aligned}
Y_h &:= H^1(0, T, V_h) \\
X_h &:= V_h \times L^2(0, T, V_h)
\end{aligned}
$$

We observe that:

$$
Y_h \subset Y, \quad \text{and} \quad X_h \subset X \tag{5}
$$

The semi-discrete problem is :

$$
\begin{cases}
\text{Find } u_h \in Y_h \quad \text{such that :} \\
\mathcal{B}(u_h, (v_{0h}, v_h)) = \ell(v_{0h}, v_h), \quad \forall (v_{0h}, v_h) \in X_h
\end{cases} \tag{6}
$$

The approximation setting is conforming due to (5).

Let $\mathcal{P}_{V_h} : L^2(\Omega) \to V_h$ be the $L^2(\Omega)$ orthogonal projection i.e

$$(z - \mathcal{P}_{V_h} z, w_h) = 0, \quad \forall w_h \in V_h$$

Proposition. (well posedness)

i) A function $u_h \in Y_h$ is a solution of (6) iff for all $w_h \in V_h$

$$\begin{aligned} (\partial_t u_h(t), w_h) + (\nabla u_h, \nabla w_h) &= (f(t), w_h) \\ u_h(0) &= \mathcal{P}_{V_h} u_0 \end{aligned} \tag{7}$$

ii) The semi-discrete problems (6) and (7) are well-posed.

Full discretisation: backward Euler in time

Let $N > 1$ be the number of time steps and let

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T$$

be the discrete times; we will denote by I_n the n-th time interval, $[t_{n-1}, t_n]$ and τ_n the length of the n-th time step,

$$\tau_n := t_n - t_{n-1} = |I_n|, \quad 1 \leq n \leq N.$$

As in the previous chapters, we let $\mathcal{T}_h$ be a simplicial mesh of the closure of the computational domain Ω. Recall that $V_{hp} = \mathcal{P}_p(\mathcal{T}_h) \cap H_0^1(\Omega)$. Let $u_0 = 0$. For all $1 \leq n \leq N$, find $u_h^n \in V_{hp}$ such that

$$\left(\frac{u_h^n - u_h^{n-1}}{\tau_n}, v_h\right) + (\nabla u_h^n, \nabla v_h) = \frac{1}{\tau_n}\int_{I_n} (f, v_h)\,\mathrm{d}t \quad \forall v_h \in V_{hp}. \tag{8}$$

Theorem

There exists a unique solution $u_h^n \in V_{hp}$ for all $1 \leq n \leq N$ solution of (8).

Error analysis in the L^2-norm

Theorem

$$\max_{1\leq n\leq N} \|u(\cdot, t^n) - u_h^n\|_{L_2(\Omega)} \leq C\left(h^2 + \tau\right),$$

where C is a positive constant independent of h and τ.

The proof can be done by decomposing the global error e_h as follows:

$$e_h^n = u(\cdot, t^n) - u_h^n = \eta^n + \xi^n,$$

where

$$\eta^n = u(\cdot, t^n) - Pu(\cdot, t^n), \quad \xi^n = Pu(\cdot, t^n) - u_h^n,$$

and for $t \in [0, T]$, $Pu(\cdot, t) \in V_h$ denotes the projection of $u(\cdot, t)$ defined by

$$a(Pu(\cdot, t), v_h) = a(u(\cdot, t), v_h) \quad \forall v_h \in V_h.$$