Lecture 4: A posteriori error analysis for Variational Inequalities

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Outline

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- local lower bounds for the error.

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Efficient a posteriori estimators are successfully used for adaptive algorithms that involve local mesh refinements.
Note that the proof of the efficiency estimate (up to possible data resp. obstacle oscillations) is an open problem (see [4] for instance).

Abstract framework

In this work we consider some constrained minimisation problems of the form:

$$\text{Find } u := \arg\min_{v\in\mathcal{K}} \left[\frac{1}{2}a(v,v) - \ell(v)\right], \tag{1}$$

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The the existence and uniqueness of the solution is a direct consequence of Stampacchia's theorem (see [1]).

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$$\begin{aligned} \mathscr{A}(u) \geq \ell \quad & \text{in } \Omega \\ Bu - \psi \geq 0 \quad & \text{in } \Omega \\ (\mathscr{A}(u) - \ell)(Bu - \psi) = 0 \quad & \text{in } \Omega \end{aligned}$$

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We then rewrite the problem using the Lagrange multiplier λ

$$\lambda = \mathscr{A}(u) - \ell. \tag{3}$$

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$$\begin{cases} \text{Find } (u,\lambda) \in V \times \Lambda \text{ such as that} \\ a(u,v) - b(v,\lambda) = \ell(v), \quad \forall v \in V, \\ b(u,\mu-\lambda) \geq g(\mu-\lambda), \qquad \forall \mu \in \Lambda. \end{cases} \tag{4}$$

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where $b(\cdot,\cdot)$ is a continuous bilinear form satisfies the following inf-sup condition:

$$\exists C_b > 0, \quad \sup_{v \in V} \frac{b(v,\xi)}{\|v\|_V} \geq C_b \|\xi\|_{V'} \quad \xi \in Q \tag{5}$$

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New approches are needed.

A new formulation

Defining on $\mathcal{H} = V \times Q$ the bilinear and linear forms

$$\mathscr{B} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R} \quad \text{and} \quad \mathscr{L} : \mathcal{H} \longrightarrow \mathbb{R}$$

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$$\mathscr{B}(w, \xi; v, \mu) = a(w, v) - b(v, \xi) - b(w, \mu), \tag{6}$$

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the variational problem can be reformulated as:

$$\begin{cases} \text{Find } (u, \lambda) \in V \times \Lambda \text{ such that } : \\ \mathscr{B}(u, \lambda; v, \mu - \lambda) \leq \mathscr{L}(v, \mu - \lambda) \quad \forall (v, \mu) \in V \times \Lambda. \end{cases} \tag{8}$$

Continuous stability

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Theorem 1 ([2])

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Then problem (8) admits a unique solution. Moreover, we have the following stability result:
For every $(v,\mu) \in V \times Q$ *there exists* $w \in V$ *such that*

$$\mathscr{B}(v,\mu;w,-\mu) \gtrsim |||(v,\mu)|||^2 \quad \text{and} \quad \|w\|_V \lesssim |||(v,\mu)||| \tag{9}$$

where,

$$|||(w,\xi)||| = \left(\|w\|_V^2 + \|\xi\|_{V'}^2\right)^{1/2}, \tag{10}$$

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Form the continuity of the bilinear form $a(\cdot, \cdot)$ it follows that

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Since q is arbitrary, we have

$$\|\mu\|_{V'} \lesssim \sup_{q \in V} \frac{b(q, \mu)}{\|q\|_V} \lesssim \|p\|_V. \tag{13}$$

Moreover, the coercivity of the bilinear form $a(\cdot,\cdot)$ gives

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Choosing $w = v - p$, noting that

$$\begin{aligned}\mathscr{B}(v,\mu; v-p,-\mu) &= a(v,v) - a(p,v) + b(p,\mu) \\ &= a(v,v) - b(v,\mu) + b(p,\mu) \\ &= \frac{1}{2}(a(v,v) + a(p,p) + a(v-p,v-p)) \\ &+ a(p,p)\end{aligned}$$

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and applying inequalities (13) and (14) proves the result.

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$$\mathscr{B}_h((u_h, \lambda_h); (v, \xi)) = \mathscr{B}((u_h, \lambda_h); (v, \xi)) - \alpha \mathcal{B}_h((u_h, \lambda_h); (v, \xi))$$
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such that:

$$\mathcal{B}_h((u, \lambda); (v, \xi)) = \mathcal{L}_h(v, \xi), \quad \forall (v, \xi) \in \mathcal{H}$$

For suitable choice of the parameter $\alpha > 0$ and for simple choice of the finite dimensional space $V_h \subset V$ and $Q_h \subset Q$,we need to get analogue results as in the continuous level, i.e.,

$\forall (v_h, \mu_h) \in V_h \times Q_h, \quad \exists w_h \in V_h$ such that

$$\mathscr{B}_h(v_h, \mu_h; w_h, -\mu_h) \gtrsim |||(v_h, \mu_h)|||^2 \quad \text{and} \quad \|w_h\|_V \lesssim |||(v_h, \mu_h)|||$$

Second order contact problems

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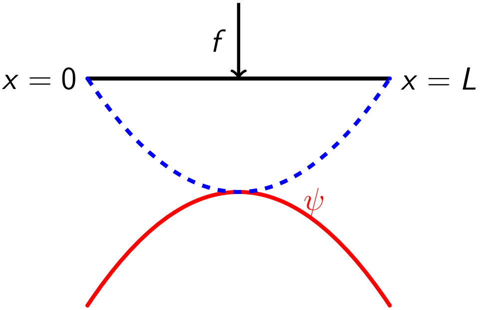

Figure: A timoshenko beam with a rigid obstacle ψ

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$$J(\eta, v) := \frac{1}{2} \int_0^L (\eta')^2 \, dx + \frac{t^{-2}}{2} \int_0^L (v' - \eta)^2 \, dx - \langle f, v \rangle$$
$$\mathcal{K} := \left\{ (\eta, v) \in H_0^1(0, L) \times H_0^1(0, L); \text{ and } v \geq \psi \right\}$$

Then if $f \in H^{-1}(0, L)$, the assumptions of our abstract framework are satisfied.

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$$\text{Finding } (\theta, w) : \arg \min_{(\eta, v)\in\mathcal{K}} J(\eta, v) \tag{15}$$

where,

$$J(\eta, v) := \frac{1}{2}\int_0^L (\eta')^2 \, dx + \frac{t^{-2}}{2}\int_0^L (v' - \eta)^2 \, dx - \langle f, v\rangle$$
$$\mathcal{K} := \left\{(\eta, v) \in H_0^1(0, L) \times H_0^1(0, L); \text{ and } v \geq \psi\right\}$$

The minimization problem consists of :

$$\text{Finding } (\theta, w) : \arg \min_{(\eta,v)\in\mathcal{K}} J(\eta, v) \tag{15}$$

where,

$$J(\eta, v) := \frac{1}{2}\int_0^L (\eta')^2 \, dx + \frac{t^{-2}}{2}\int_0^L (v' - \eta)^2 \, dx - \langle f, v\rangle$$
$$\mathcal{K} := \left\{(\eta, v) \in H_0^1(0, L) \times H_0^1(0, L);\ \text{and } v \geq \psi\right\}$$

Then if $f \in H^{-1}(0, L)$, the assumptions of our abstract framework are satisfied.

Then we have the following result:

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Theorem 2

Problem (15) admits a unique solution.

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Problem (15) admits a unique solution. Moreover, the minimizer (θ, w) of problem (15) satisfies

$$\begin{cases} \textit{Find } (\theta, w) \in \mathcal{K} \textit{ such that } \forall (\eta, v) \in \mathcal{K} \\ \displaystyle\int_0^L \theta'(\eta' - \theta')\, dx + t^{-2} \int_0^L (w' - \theta)((v - w)' - (\eta - \theta))\, dx \\ \qquad\qquad\qquad\qquad \geq \langle f, v - w \rangle \end{cases}$$

Using suitable choice of test functions we can easily show that the complementarity system reads:

$$\left.\begin{aligned} -\theta'' - t^{-2}(w' - \theta) &= 0 \\ -t^{-2}(w' - \theta)' &\geq f \\ w - \psi &\geq 0 \\ (-t^{2}(w' - \theta)' - f)(w - \psi) &= 0 \end{aligned}\right\} \quad \text{a.e in } \Omega \tag{16}$$

$$\begin{cases} \theta(0) = w(0) = 0 \\ w(L) = \theta(L) = 0 \end{cases} \tag{17}$$

Theorem 3

If f and g are in $L^2(0, L)$, then $(\theta, w) \in (H^2(0, L))^2$.

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Remark 1

If $g = 0$ (or $g \in H^1(0,L)$), then we can expect that $\theta \in H^3(0,L)$, which mean that θ is more regular then w, which can not be happen in the equality cas. But this is not surprising because w is constrained whereas θ is not. The regularity of w is limited even when f and g are very smooth, it cannot exceed $C^{1,1}$.

Mixed formulation

By introducing:

Mixed formulation

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$$\lambda = -t^{-2}(w' - \theta)' - f$$

the complementarity system reads:

$$\left.\begin{aligned} -\theta'' - t^{-2}(w' - \theta) = 0 \\ -t^{-2}((w' - \theta)' - \lambda = f \\ \lambda \geq 0 \\ \lambda(w - \psi) = 0 \end{aligned}\right\} \quad \text{a.e in } \Omega \tag{18}$$

$$\begin{cases} \theta(0) = w(0) = 0 \\ \theta(L) = w(L) = 0 \end{cases} \tag{19}$$

We define,

$$V = H_0^1(0, L) \times H_0^1(0, L), \quad Q = H^{-1}(0, L)$$

and let Λ be the space defined by,

$$\Lambda := \{\mu \in Q, \langle \mu, \varphi \rangle \geq 0, \quad \forall \varphi \geq 0, \varphi \in C_0^\infty(0, L)\}$$

and we consider the following mixed variational formulation

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$$\begin{cases} \text{Find } (\theta, w, \lambda) \in V \times \Lambda \\ a((\theta, w), (\eta, v)) + b((\eta, v), \lambda) = F(\eta, v) \quad \forall (\eta, v) \in V \\ b((\theta, w), \mu - \lambda) \leq G(\mu - \lambda), \qquad \forall \mu \in \Lambda \end{cases} \tag{20}$$

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where,

$$a((\theta, w); (\eta, v)) = \int_0^L \theta' \eta' \, dx + t^{-2} \int_0^L (w' - \theta)(v' - \eta) \, dx,$$

$$b((\eta, v), \lambda) = -\langle v, \lambda \rangle, \quad F(\eta, v) = \langle f, v \rangle, \quad G(\mu) = -\langle \psi, \mu \rangle$$

Consistent FE Discretization

Let us consider a finite dimensional space $V_h \subset H_0^1(0,L)$, $\eta_h \subset H_0^1(0,L)$ and $Q_h \subset L^2(0,L)$. Moreover, we introduce the closed convex set

$$\Lambda_h := \{\mu_h \in Q_h; \quad \mu_h \geq 0\}$$

Then we introduce the following bilinear and linear forms:

$$\begin{aligned}\mathcal{S}_h((\theta,w,\lambda);(\eta,v,\xi)) =& h^2[(\theta'' + t^{-2}(w'-\theta), \eta'' + t^{-2}(v'-\eta) \\ & + (t^{-2}(w''-\theta') + \lambda, t^{-2}(v''-\eta') + \xi)] \\ \mathcal{L}_h(\eta,v,\xi) =& h^2\left(f, t^{-2}(v''-\eta') + \xi\right)\end{aligned}$$

and for suitable choice of the parameter $\alpha > 0$, we define,

$$\begin{aligned}\mathscr{B}_h((\theta,w,\lambda);(\eta,v,\xi)) &= \mathscr{B}((\theta,w,\lambda);(\eta,v,\xi)) \\ &\quad - \alpha\mathcal{B}_h((\theta,w,\lambda);(\eta,v,\xi)) \\ \mathscr{L}_h(\eta,v,\xi) &= \mathscr{L}(\eta,v,\xi) - \alpha\mathcal{L}_h(\eta,v,\xi)\end{aligned}$$

Finally, we consider the following problem:

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$$\begin{cases} \text{Find } (\theta_h, w_h, \lambda_h) \in \eta_h \times V_h \times \Lambda_h \quad \text{such that} \\ \forall(\eta_h, v_h, \mu_h) \in \eta_h \times V_h \times \Lambda_h \\ \mathscr{B}_h((\theta_h, w_h, \lambda_h); (\eta_h, v_h, \mu_h - \lambda_h)) \leq \mathscr{L}_h(\eta_h, v_h, \mu_h - \lambda_h) \end{cases} \tag{21}$$

Then we can choose the finite element spaces:

$$\left.\begin{cases} V_h = \eta_h = \mathbb{P}_{k+2} \\ Q_h = \mathbb{P}_k \end{cases}\right| \quad k \geq 0$$

to get the discrete stability:

$\forall(v_h, \mu_h) \in V_h \times Q_h, \quad \exists w_h \in V_h$ such that

$\mathscr{B}_h(v_h, \mu_h; w_h, -\mu_h) \gtrsim |||(v_h, \mu_h)|||^2$ and $\|w_h\|_V \lesssim |||(v_h, \mu_h)|||$

The obstacle problem of an elastic membrane consists of

$$\left\{ \text{Finding } u := \arg\min_{v \in \mathcal{K}} J(v) \right. \tag{22}$$

where,

$$J(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \langle f, v \rangle$$
$$\mathcal{K} := \{ v \in H_0^1(\Omega); \quad v \geq \psi \text{ a.e in } \Omega \}$$

Then, the primal problem reads:

$$\begin{cases} \text{Find } u \in \mathcal{K} \text{ such that} \\ a(u, v-u) \geq \langle f, v-u \rangle, \quad \forall v \in \mathcal{K} \end{cases} \tag{23}$$

with,

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx$$

The strong formulation reads:

$$
\begin{aligned}
-\Delta u - f &\geq 0, && \text{dans } \Omega \\
u &\geq \psi, && \text{dans } \Omega \\
(\Delta u + f)(u - \psi) &= 0, && \text{dans } \Omega \\
u &= 0, && \text{sur } \Gamma
\end{aligned}
\tag{24}
$$

We introduce the Lagrange multiplier $\lambda = \Delta u + f$, then the problem reads:

$$
\begin{aligned}
-\Delta u - \lambda &= f, && \text{dans } \Omega \\
u &\geq \psi, && \text{dans } \Omega \\
\lambda &\geq 0, && \text{dans } \Omega \\
\lambda(u - \psi) &= 0, && \text{dans } \Omega \\
u &= 0, && \text{sur } \Gamma
\end{aligned}
\tag{25}
$$

The Lagrange multiplier belongs to the space

$$Q = H^{-1}(\Omega).$$

We introduce, the set:

$$\Lambda = \{\mu \in H^{-1}(\Omega) | \quad \langle \mu, v \rangle_{H^{-1}, H_0^1}, \geq 0, \forall v \in \boldsymbol{V}, v \geq 0 \text{ a.e in } \Omega\}$$

The mixed formulation reads:

$$\begin{cases} \quad \text{Trouver } (u, \lambda) \in \boldsymbol{V} \times \Lambda \text{ such that} \\ a(u, v) + b(v, \lambda) = \ell(v), \qquad \forall v \in \boldsymbol{V} \\ \qquad b(u, \mu - \lambda) \leq g(\mu - \lambda), \quad \forall \mu \in \Lambda \end{cases} \tag{26}$$

with,

$$b(v, \mu) = -\langle \lambda, v \rangle_{H^{-1}, H_0^1}, \quad G(\mu) = -\langle \psi, \mu \rangle$$

A compact formulation

Defining on $\mathcal{H} = V \times Q$ the bilinear and linear forms

$$\mathscr{B} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathbb{R} \quad \text{and} \quad \mathscr{L} : \mathcal{H} \longrightarrow \mathbb{R}$$

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through

$$\mathscr{B}(w, \xi; v, \mu) = a(w, v) - b(v, \xi) - b(w, \mu), \tag{27}$$

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the variational problem can be reformulated as:

$$\begin{cases} \text{Find } (u, \lambda) \in V \times \Lambda \text{ such that } : \\ \mathscr{B}(u, \lambda; v, \mu - \lambda) \leq \mathscr{L}(v, \mu - \lambda) \quad \forall (v, \mu) \in V \times \Lambda. \end{cases} \tag{29}$$

$$\mathscr{B}_h((u_h,\lambda_h);(v,\xi)) = \mathscr{B}((u_h,\lambda_h);(v,\xi)) - \alpha\mathcal{B}_h((u_h,\lambda_h);(v,\xi))$$
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with

$$\mathcal{B}_h((w,\xi);(v,\mu)) := \sum_{T\in\mathcal{T}_h} h_T^2(-\Delta w - \xi, -\Delta v - \mu)_T$$
$$\mathcal{L}_h(v,\mu) := \sum_{T\in\mathcal{T}_h} h_T^2(f, -\Delta v - \mu)_T$$

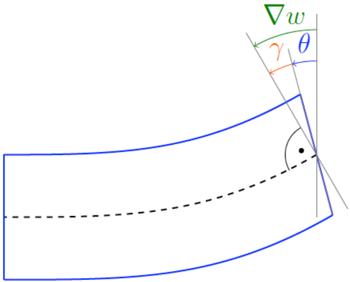
Note that with the assumption $f \in L^2(\Omega)$ it holds that $\Delta u + \lambda \in L^2(\Omega)$ even if $\Delta u \notin L^2(\Omega)$ and $\lambda \notin L^2(\Omega)$. Hence, it holds that:

$$\mathcal{B}_h(u,\lambda; v_h,\mu_h) = \mathcal{L}_h(v_h,\mu_h), \quad \forall (v_h,\mu_h) \in V_h \times \Lambda_h \tag{30}$$

Reissner-Mindlin plate

For a clamped Reissner-Mindlin plate we look for a rotation $\boldsymbol{\theta}$ and displacement w in the set $\mathcal{K}$ defined by:

$$\mathcal{K} := \{(\boldsymbol{\eta}, v) \in V := \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega); \quad w \geq \psi\}$$



Reissner-Mindlin plate

The problem consists of

$$\text{Finding } (\boldsymbol{\theta}, w) =: \underset{(\boldsymbol{\eta}, v) \in \mathcal{K}}{\arg\min} \; J(\boldsymbol{\eta}, v) \tag{31}$$

where,

$$J(\boldsymbol{\theta}, w) = \frac{1}{2} \int_{\Omega} \mathbb{C}\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\theta}) \; dx + \frac{\lambda t^{-2}}{2} \int_{\Omega} |\nabla w - \boldsymbol{\theta}|^2 \; dx - \int_{\Omega} fw \; dx$$

where the matrix $\mathbb{C}$ and the scalar λ depend on plate materiel, t is the thickness of the plate and f represents the loading We recall the

$$\varepsilon(\boldsymbol{\theta}) = \begin{pmatrix} \partial_1 \theta_1 & \dfrac{\partial_1 \theta_2 + \partial_2 \theta_1}{2} \\ \dfrac{\partial_2 \theta_1 + \partial_1 \theta_2}{2} & \partial_2 \theta_2 \end{pmatrix}$$

Fourth order contact problems

Bernoulli's beam

Now we consider a clamped Bernoulli's beam posed over an obstacle ψ, the equilibrium problem reads :

$$\text{Find } u := \underset{v \in \mathcal{K}_B}{\operatorname{argmin}} J(v)$$

where,

$$J(v) := \frac{1}{2} \int_0^L (v'')^2 \, dx - \langle f, v \rangle$$

$$\mathcal{K}_B := \left\{ v \in V := H_0^2(0, L); \; v \geq \psi \right\}$$

Then,

$$a(u, v) := \int_0^L u'' v'' \, dx, \quad b(v, \xi) = \langle \xi, v \rangle$$

$$\ell(v) := \langle f, v \rangle, \quad g(\mu) = (\psi, \mu)$$

The complementarity system reads:

$$\left.\begin{aligned} w^{(4)} - \lambda = f \\ \lambda \geq 0 \\ \lambda(w - \psi) = 0 \end{aligned}\right\} \quad \text{a.e in } \Omega \tag{32}$$

$$\begin{cases} w(0) = w'(0) = 0 \\ w(L) = w'(L) = 0 \end{cases} \tag{33}$$

Remark 2

Under appropriate smoothness assumptions, the solution to the Bernoulli's problem over a rigid obstacle is in H^3 but it cannot belong to H^4. The exact solutions given in [5] seem to indicate that the smoothness threshold is $C^{2,1}$ or $H^{7/2-\epsilon}, \epsilon > 0$.

Compact form

We define,

$$
\begin{aligned}
\mathcal{H} :=& H_0^2(0,L) \times H^{-2}(0,L) \\
\Lambda :=& \{\mu \in H^{-2}(0,L); \mu \geq 0; \quad \text{a.e in } (0,L)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{B}(w,\mu;(v,\xi)) &= a(w,v) - b(v,\mu) - b(w,\xi) \\
\mathscr{L}(v,\xi) &= (f,v) - (\psi,\xi)
\end{aligned}
$$

the variational problem can be reformulated as:

$$
\begin{cases}
\text{Find } (u,\lambda) \in V \times \Lambda \text{ such that } : \\
\mathscr{B}(u,\lambda; v, \mu - \lambda) \leq \mathscr{L}(v, \mu - \lambda) \quad \forall (v,\mu) \in V \times \Lambda.
\end{cases}
\tag{34}
$$

VC Discretization

Let $\mathscr{C}_h$ be a discretization of $[0, L]$, we consider the finite element subspaces

$$V_h \subset H_0^2(0, L), \quad Q_h \subset H^{-2}(0, L) \tag{35}$$

Moreover, we define

$$\Lambda_h = \{\mu_h \in Q_h : \mu_h \geq 0 \text{ in } (0, L)\} \subset \Lambda. \tag{36}$$

Let us introduce bilinear and linear forms $\mathscr{B}_h$ and $\mathscr{L}_h$ by

$$\mathscr{B}_h(w, \xi; v, \mu) = \mathscr{B}(w, \xi; v, \mu) - \alpha \mathcal{S}_h(w, \xi; v, \mu)$$

$$\mathcal{S}_h(w, \xi; v, \mu) = \sum_{I \in \mathscr{C}_h} h_I^4 (w^{(4)} - \xi, v^{(4)} - \mu)_I$$

$$\mathscr{L}_h(v, \mu) = \mathscr{L}(v, \mu) - \alpha \sum_{I \in \mathscr{C}_h} h_I^4 (f, v^{(4)} - \mu)_I, \quad \alpha > 0$$

Discrete problem

$$\begin{cases} \text{Find } (u_h, \lambda_h) \in V_h \times \Lambda_h \text{ such that } : \\ \mathscr{B}_h(u_h, \lambda_h; v_h, \mu_h - \lambda_h) \leq \mathscr{L}_h(v, \mu_h - \lambda_h) \quad \forall (v_h, \mu_h) \in V_h \times \Lambda_h. \end{cases}$$

Let us introduce the following norm

$$|||(w_h, \xi_h)|||_h^2 = \|w_h\|_2^2 + \|\xi_h\|_{-2}^2 \tag{37}$$

Theorem 4

Then for all $(v_h, \mu_h) \in V_h \times Q_h$ there exists $w_h \in v_h$ such that

$$\mathscr{B}_h(v_h, \mu_h; w_h, -\mu_h) \gtrsim |||(v_h, \mu_h)|||_h^2 \tag{38}$$

$$|||(w_h, -\mu_h|||_h) \lesssim |||(v_h, \mu_h)|||_h \tag{39}$$

(A priori estimate)

Theorem 5

It holds that

$$
\begin{aligned}
&|||(u - u_h, \lambda - \lambda_h)||| \\
&\quad \lesssim \inf_{v_h \in V_h, \mu_h \in \Lambda_h} \left(|||(u - v_h, \lambda - \mu_h)||| + \sqrt{\langle u - \psi, \mu_h \rangle} \right) + osc(f).
\end{aligned}
$$

(A priori estimate)

Theorem 5

It holds that

$$
\begin{aligned}
&|||(u-u_h,\lambda-\lambda_h)||| \\
&\quad \lesssim \inf_{v_h\in V_h,\mu_h\in\Lambda_h}\left(|||(u-v_h,\lambda-\mu_h)||| + \sqrt{\langle u-\psi,\mu_h\rangle}\right) + osc(f).
\end{aligned}
$$

$$
= \inf_{v_h\in V_h,\mu_h\in\Lambda_h}\left(|||(u-v_h,\lambda-\mu_h)||| + \sqrt{\langle u-\psi,\mu_h-\lambda\rangle}\right) + osc(f).
$$

$$
\sim h^{\alpha}, \quad \alpha > 0
$$

For the Kircchoff plate model, we define:

$$\boldsymbol{K}(w) = -\boldsymbol{\varepsilon}(\nabla w),$$
$$\boldsymbol{M}(w) = \frac{t^3}{12} C\boldsymbol{K}(w).$$

The obstacle problem of a clamped Kircchoff plate reads:

$$\text{Find } u = \underset{v\in\mathcal{K}}{\arg\min}\left[\frac{1}{2}a(v,v) - \ell(v)\right], \tag{40}$$

with

$$a(w,v) = \int_\Omega \boldsymbol{M}(w) : \boldsymbol{K}(v)dx,$$
$$\ell(v) = \int_\Omega fv \; dx$$
$$\mathcal{K} = \{v \in H_0^2(\Omega) : v \geq \psi \text{ in } \Omega\}.$$

The strong form is thus : Find u and λ such that

$$\mathscr{A}(u) - \lambda = \ell \quad \text{in } \Omega \tag{41}$$

$$\lambda \geq 0 \quad \text{in } \Omega \tag{42}$$

$$\lambda (u - \psi) = 0 \quad \text{in } \Omega \tag{43}$$

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{44}$$

with the biharmonic operator $\mathscr{A}(u)$ given by

$$\mathscr{A}(u) := D\Delta^2 u \tag{45}$$

where D stands for the bending stiffness defined through

$$D = \frac{Et^3}{12}(1 - \nu^2) \tag{46}$$

Consistent FE discretization

Let $\mathcal{T}_h$ be a conforming shape regular triangulation of Ω which we assume to be polygonal. The finite element subspaces are

$$V_h \subset V, \quad Q_h \subset Q \tag{47}$$

Moreover, we define

$$\Lambda_h = \{\mu_h \in Q_h : \mu_h \geq 0 \text{ in } \Omega\} \subset \Lambda. \tag{48}$$

Let us introduce bilinear and linear forms $\mathscr{B}_h$ and $\mathscr{L}_h$ by

$$\mathscr{B}_h(w,\xi;v,\mu) = \mathscr{B}(w,\xi;v,\mu) - \alpha\mathcal{S}_h(w,\xi;v,\mu)$$

$$\mathcal{S}_h(w,\xi;v,\mu) = \sum_{T\in\mathcal{T}_h} h_T^4(\mathscr{A}(w)-\xi,\mathscr{A}(v)-\mu)_T$$

$$\mathscr{L}_h(v,\mu) = \mathscr{L}(v,\mu) - \alpha\sum_{T\in\mathcal{T}_h} h_T^4(f,\mathscr{A}(v)-\mu)_T, \quad \alpha > 0$$

We then consider the following proble: **Problem 2** Find $(u_h, \lambda_h) \in V_h \times \Lambda_h$ such that

$$\mathscr{B}(u_h, \lambda_h; v_h, \mu_h - \lambda_h) \leq \mathscr{L}(v_h, \mu_h - \lambda_h) \quad \forall (v_h, \mu_h) \in V_h \times \Lambda_h$$

Then for the conforming finite element spaces :

$$V_h = \{v_h \in H_0^2(0, L), \quad v_{h|T} \in \mathbb{P}^5(T), \forall T \in \mathcal{T}_h\}$$
$$Q_h = \{\mu_h \in Q; \quad \mu_{h|T} \in \mathbb{P}^k(T), \forall T \in \mathcal{T}_h, \ k \geq 0\}$$

Then we obtain:

- The analogue discret stability result as in Theorem 4.
- The analogue a priori error estimate as in Theorem 5.

A posteriori error estimate

First we recall the following integration by parts formula, valid in any domain $R \subset \Omega$

$$a_R(w,v) = \int_R \mathscr{A}(w)v dx - \int_{\partial R} Q_n(w)v ds - \int_{\partial R} (M(w)\frac{\partial v}{\partial n} + M_{ns}(w)\frac{\partial v}{\partial s})\ ds.$$

where we have used the shorthand notation

$$a_R(w,v) = \int_R \boldsymbol{M}(w) : \boldsymbol{K}(v) dx,$$

A posteriori error estimate

Now we define the normal shear force and the normal and twisting moments through

$$\begin{aligned} Q_n(w) =& Q(w) \cdot n, \quad M_{nn}(w) = n \cdot M(w) n, \\ M_{ns}(w) =& M_{sn}(w) = s \cdot M(w) n, \end{aligned}$$

with n and s denoting the normal and tangential directions at R. integrating by parts on a smooth $S \subset R$ we get

$$\int_S Q_w v ds - \int_s M_{ns}(w) \frac{\partial v}{\partial s} ds = \int_S v_n(w) ds - |_p^q M_{ns}(w) v, \tag{49}$$

where p and q are the endpoints of S and the quantity

$$V_n(w) = Q_n(w) + \frac{\partial M_{ns}(w)}{\partial s} \tag{50}$$

Denote by $\omega_E = T_1 \bigcup T_2$ the pair of triangles sharing an edge E and define jumps in the normal moment and the shear force over E through

$$\begin{aligned} [\![M_{nn}(v)]\!]_E &= M_{nn}(v) - M_{n'n'}(v) \\ [\![V_n(v)]\!]_E &= V_n(v) + V_{n'}(v) \end{aligned}$$

where n and n' stand for the outward normals to T_1 and T_2, respectively.

To derive a posteriori error bounds, we define the local residual estimators

$$\eta_T^2 = h_T^4 \|\mathscr{A}(u_h) - \lambda_h - f\|_{0,T}^2, \tag{51}$$

$$\eta_E^2 = h_E^3 \|[\![V_n(u_h)]\!]\|_{0,E}^2 + h_E \|[\![M_{nn}(u_h)]\!]\|_{0,E}^2, \tag{52}$$

and the corresponding global estimator

$$\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2 + \sum_{E \in \mathscr{E}_h^I} \eta_E^2, \tag{53}$$

where $\mathscr{E}_h^I$ denotes the set of interior edges in the mesh, An additional global estimator S, due to the unknown location of the contact boundary, is defined through

$$S^2 = (u_h - \psi)_+ + \sum_{T \in \mathcal{T}_h} \frac{1}{h_T^4} \|(\psi - u_h)_+\|_{0,T}^2 \tag{54}$$

where $u_+ = \max(u, 0)$ denotes the positive part of u.

The following a posteriori estimate holds:

Theorem 6

$$|||(u - u_h, \lambda - \lambda_h)||| \lesssim \eta + S$$

Remark 3

the upper bound cannot be established as elegantly as for the second order problem, since the positive part function is not in $H^2(\Omega)$.

Theorem 7

$$\eta \lesssim |||(u - u_h, \lambda - \lambda_h)|||. \tag{55}$$

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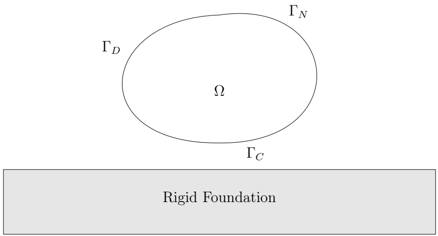
(55) can be proved with the help of the following saturation assumption: There exists $\beta < 1$ such that:

$$|||(u - u_{h/2}, \lambda - \lambda_{h/2})|||_{h/2} \le \beta |||(u - u_h, \lambda - \lambda_h)|||_h$$

which itself an open problem.

Numerical Tests

Signorini problem



Signorini problem

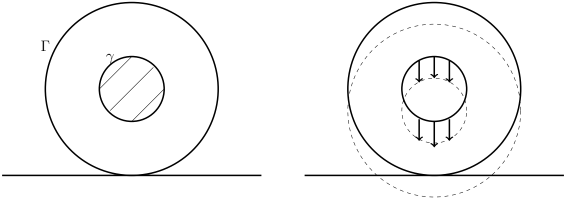
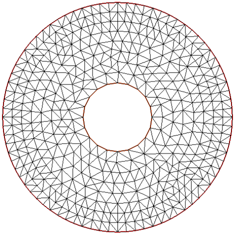
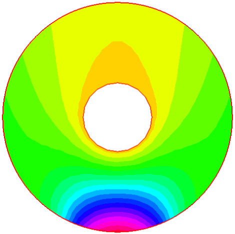


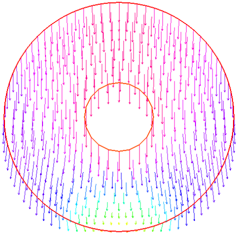
FIGURE 1 – The domain Ω



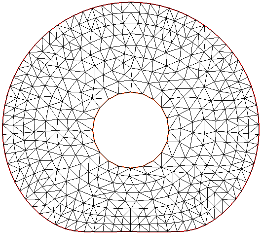
(a) The mesh



(b) Isovalues



(c) Displacements



(d) Deformation

Timoshenko's beam

Now we present numerical tests for Timoshenko's beam:

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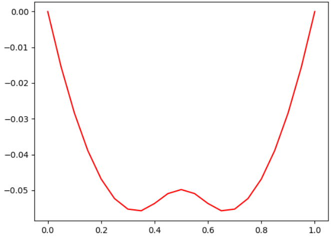
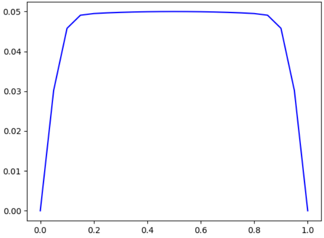
For both tests we use the finite element $P_2 - P_1 - P_0$ and consider very small thickness $t = 0.001$ in order to test the method agains the locking phenomena.

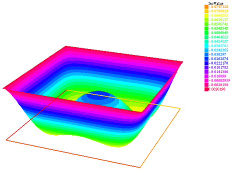


(a) with $w \geq \psi_1$

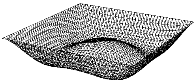


(b) with $w \leq \psi_2$

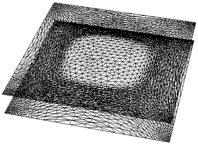
Kircchoff plate



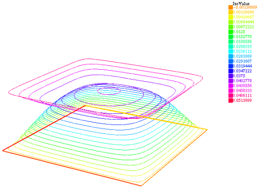
(a) isovalues



(b) deformed



(c) The obstacle



(d) the constraint

The finite element method leads to stable discrete problems for very simple choice of the finite element subspaces even for very small values of thickness t.

The finite element method leads to stable discrete problems for very simple choice of the finite element subspaces even for very small values of thickness t. Therefore, at least for the considered problem, it is locking-free. It allows us to use different and various approaches to solve a constrained minimization problem as it reformulates the problem as a standard variational inequality.

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