# Interpolation Error and Convergence

# Main objective

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$\forall K \in \mathcal{T}_h, \forall v \in H^{\ell+1}(K)$

$$|v - \tau_h(v)|_{m,K} \leq c\, h_K^{\ell+1-m} |v|_{\ell+1,K}, \quad 0 \leq m \leq \ell + 1$$

where, $\tau_h$ is the interpolation operator that maps $H^{\ell+1}$ into the Finite elements space.

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where, $\tau_h$ is the interpolation operator that maps $H^{\ell+1}$ into the Finite elements space.
This inequality can be used to prove the convergence of the finite element solution when the mesh size $h$ goes to zero.

# Preliminaries:

**Theorem. (Poincaré-Wirtinger Inequality)**

Let $\Omega \subset \mathbb{R}^d$ be a bounded, connected, open subset with a Lipschitz boundary, and let $1 \leq p < \infty$.
For any function $u \in W^{1,p}(\Omega)$ such that $u$ has zero mean, i.e.,

$$\int_\Omega u \, dx = 0,$$

there exists a constant $C$ depending only on $\Omega$ and $p$ such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}.$$

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**Theorem. (Rellich-Kondrachov)**

Let $\Omega \subset \mathbb{R}^d$ be a bounded open subset with a Lipschitz boundary. For $1 \leq p < \infty$. If $\{u_k\}$ is a bounded sequence in $W^{1,p}(\Omega)$, then there exists a subsequence $\{u_{k_j}\}$ that converges in $L^q(\Omega)$ where $1/p + 1/q = 1$.

# Interpolation in one dimension: $d = 1$

For : a fixed integer $m > 0$ and $h = 1/N$, we consider the spaces of Lagrange finite elements $\mathbb{P}_m$:

$$V_h^m := \left\{ f \in C^0([0,1]) \mid f|_{[jh,(j+1)h]} \in \mathbb{P}_m, j = 0, \ldots, N-1 \right\}$$

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- $\mathbb{P}_m$ is the set of polynomials of degree less than or equal to $m$.
- The polynomials $\psi_i$ are defined by their explicit expression:

$$\psi_i(y) = \frac{\prod_{j \neq i}(y - y_j)}{\prod_{j \neq i}(y_i - y_j)}.$$

where, for all $j$, $0 \leq j \leq m$; $y_j = j/m$

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We can easily construct a basis $(\Phi_k)$ of $V_h^m$ using the basis functions $\psi_j$.

# Basis Functions $\Phi_k(x)$

We associate with the space $V_h^m$ the set of degrees of freedom

$$x_k = k/(Nm), \text{ where } k = 0, \ldots, Nm.$$

For each $k = 0, \ldots, Nm$, there exists a unique pair of integers $p$ and $q$ such that: $0 \le p \le N$ and $0 \le q < m$, and $k = pm + q$.

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If $q = 0$,

$$\Phi_k(x) = \begin{cases} 0 & \text{if } x \leq ph \\ \psi_m((x-(p-1)h)/h) & \text{if } x \in [(p-1)h, ph] \\ \psi_0((x-ph)/h) & \text{if } x \in [ph, (p+1)h] \\ 0 & \text{if } x \geq (p+1)h \end{cases}$$

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If $q \neq 0$,

$$\Phi_k(x) = \begin{cases} 0 & \text{if } x \leq ph \\ \psi_q((x-ph)/h) & \text{if } x \in [ph, (p+1)h] \\ 0 & \text{if } x \geq (p+1)h \end{cases}$$

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Finally, we define the interpolation operator $\tau_h$ from $H^1(0,1)$ into $\tau_h$ by

$$\tau_h u = \sum_{k=0}^{Nm} u(x_k)\Phi_k.$$

# Questions:

1. Show that for $1 \leq n \leq m+1$, there exists a constant $C$ such that for any $u \in H^n(]0,1[)$:

$$\|r_1 u - u\|_{H^1} \leq C \|u^{(n)}\|_{L^2}.$$

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2. Deduce that for $1 \leq n \leq m+1$, there exists a constant $C$ such that for any $u \in H^n(]0,1[)$:
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Similarly, show that there exists a constant $C$ such that for any $u \in H^n(]0,1[)$:
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3. Deduce an error estimate for the Lagrange finite element method $\mathbb{P}_m$ applied to the problem:
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# Answer for Question 1:

<u>Hint:</u>

- Let $\Pi_{n-1}$ be the $L^2$ orthogonal projection onto the set of polynomials of degree at most $n-1$. Show that for any $u \in H^n(]0,1[)$ and $n \leq m+1$, we have:

$$u - r_1 u = v - r_1 v,$$

where $v = u - \Pi_{n-1} u$.

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Let $\Pi_{n-1}$ be the orthogonal $L^2$ projection onto the set of polynomials of degree at most $n-1$.
Since

$$\tau_1 \Pi_{n-1} u = \Pi_{n-1} u \quad \text{for} \quad n \leq m+1,$$

we have

$$u - \tau_1 u = v - \tau_1 v, \text{ where } v = u - \Pi_{n-1} u.$$

# Answer for Question 1

By the definition of $\Pi_{n-1}$, if $u \in H^n(]0,1[)$ then $v \in \tilde{H}^n(]0,1[)$, where $\tilde{H}^n(]0,1[)$ is the set of functions in $H^n(]0,1[)$ that are $L^2$-orthogonal to polynomials of degree $\leq n-1$, i.e,

$$\tilde{H}^n(]0,1[) := \left\{ v \in H^n(]0,1[); \quad \int_0^1 v(x) p_{n-1}(x)\, dx = 0, \quad \forall p_n \in \mathbb{P}_{n-1} \right\}$$

The inequality for $k = 0$ is a Poincaré-Wirtinger inequality.
Suppose that there exists a constant $C > 0$ such that

$$\left\| v^{(k-1)} \right\|_{L^2} \leq C \left\| v^{(k)} \right\|_{L^2} \quad \forall v \in \tilde{H}^n(]0,1[).$$

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Let us show that there exists $\tilde{C} > 0$ such that

$$\left\| v^{(k)} \right\|_{L^2} \leq \tilde{C} \left\| v^{(k+1)} \right\|_{L^2} \quad \forall v \in \tilde{H}^n(]0,1[).$$

If this were not the case, then there would exist a sequence $v_\ell \in \tilde{H}^n(]0,1[)$ such that

$$\left\| v_\ell^{(k)} \right\|_{L^2} = 1 \text{ and } \left\| v_\ell^{(k+1)} \right\|_{L^2} \to 0 \text{ as } \ell \to \infty.$$

By the recurrence hypothesis, $(v_\ell)$ is thus a bounded sequence in $H^{k+1}(]0,1[)$.

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By the recurrence hypothesis, $(v_\ell)$ is thus a bounded sequence in $H^{k+1}(]0,1[)$.
The Rellich Lemma tells us that there exists a subsequence (which we denote by $(v_\ell)$) converging in $H^k(]0,1[)$.
Since $\left\| v_\ell^{(k+1)} \right\|_{L^2} \to 0$, we deduce that $(v_\ell)$ is a Cauchy sequence, and therefore convergent, in $H^{k+1}(]0,1[)$ to a function $v \in H^{k+1}(]0,1[)$.
The limit satisfies $v^{(k+1)} = 0$ and is also $L^2$-orthogonal to polynomials of degree $\leq n-1$. Thus, $v = 0$, which contradicts $\left\| v^{(k)} \right\|_{L^2} = 1$.

- We deduce that for $1 \leq n \leq m+1$,

$$\|v\|_{H^1} \leq C \left\| v^{(n)} \right\|_{L^2}.$$

Since $H^1(]0,1[)$ is continuously embedded in $C^0([0,1])$, we conclude the existence of a constant $\tilde{C}$ such that

$$\|\tau_1 v\|_{H^1} \leq \tilde{C}\|v\|_{H^1}.$$

The desired inequality is then obtained through a simple triangular inequality.

# Answer for Question 2

For all $v \in H^1(]0,1[)$, we have

$$\|\tau_h v - v\|^2_{L^2(0,1)} = \sum_{j=0}^{N-1} \int_{j/N}^{(j+1)/N} |\tau_h v(x) - v(x)|^2 \, dx.$$

For each $j \in \{0, \ldots, N-1\}$, let

$$w_j(x) = v((x+j)/N)$$

then

$$\tau_1 w_j(x) = (\tau_h v)\left((x+j)/N\right)$$

and by a change of variable,

$$\int_{j/N}^{(j+1)/N} |\tau_h v(x) - v(x)|^2 \, dx = h \int_0^1 |\tau_1 w_j - w_j|^2 \, dx.$$

# Answer for Question 2

Thus,

$$\|\tau_h v - v\|^2_{L^2(0,1)} = h \sum_{j=0}^{N-1} \|\tau_1 w_j - w_j\|^2_{L^2(0,1)} .$$

From the inequality established in question 2, we deduce that

$$\|\tau_h v - v\|^2_{L^2(0,1)} \leq Ch \sum_{j=0}^{N-1} \int_0^1 \left| w_j^{(n)} \right|^2 dx.$$

Combining this inequality with the equation, we obtain the desired estimate.

## Answer for Question 3

By Céa's Lemma we have

$$\|u - u_h\|_{H^1} \leq Cte \inf_{v_h \in V_h^m} \|u - v_h\|_{H^1}$$

Take

$$v_h = \tau_h u$$

then,

$$\|u - u_h\|_{H^1} \leq Cteh^m |u|_{m+1}$$