Exercise: Nonlinear Laplacian Problem

Let Ω be a bounded open set with a sufficiently smooth boundary Γ, and let $f \in L^{p'}(\Omega)$, where $p' > 1$. We consider the following problem:

$$\begin{cases} A(u) = -\text{div}\left(|\nabla u|^{p-2}\nabla u\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \tag{1}$$

where p is the harmonic conjugate of p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. We recall that the operator A satisfies:

$$A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) \tag{2}$$

$$\langle A(u) - A(v), u - v\rangle \geq \alpha \frac{\|u - v\|^2}{(\|u\| + \|v\|)^{2-p}}, \qquad \text{if } p \in (1, 2] \tag{3}$$

$$\langle A(u) - A(v), u - v\rangle \geq \beta \|u - v\|^2, \qquad \text{if } p \in [2, \infty) \tag{4}$$

for some $\alpha > 0$ and $\beta > 0$.

Question 1

Write the variational formulation of the boundary value problem (1).

The problem can be written as:

$$\begin{cases} \text{Find } u \in V = W_0^{1,p}(\Omega), \\ \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \int_\Omega f v, \quad \forall v \in V. \end{cases} \tag{5}$$

Question 2

Prove that the variational problem admits a unique solution $u \in V$.

The operator A is Lipschitz-continuous and strongly monotone. Hence the existence and the uniquness of the solution follows by standard fixed point argument.

Question 3

Let $V_h \subset V$ be a finite element subspace associated with a regular triangulation $\mathcal{T}_h$. Write the discrete problem.

The discrete problem is:

$$\begin{cases} \text{Find } u_h \in V_h, \\ \displaystyle\int_\Omega |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla v_h = \int_\Omega f v_h, \quad \forall v_h \in V_h. \end{cases} \tag{6}$$

Question 4

Define the residual $R : W_0^{1,p}(\Omega) \to \mathbb{R}$ by

$$\langle R, v\rangle = \int_\Omega |\nabla u_h|^{p-2}\nabla u_h \cdot \nabla v - \int_\Omega f v.$$

Show that the residual R is an element of $W^{-1,p'}(\Omega)$.

We have $|\nabla u_h|^{p-2}\nabla u_h \in L^\infty(\Omega) \subset L^q(\Omega), \forall q > 1$, and $f \in L^{p'}(\Omega)$. Then by Young inequality,

$$\exists C > 0, \text{ s.t} \quad |\langle R, v\rangle| \leq C\|v\|_{1,p}, \quad \forall v \in W_0^{1,p}(\Omega). \tag{7}$$

Question 5

Show the a posteriori estimate

$$\|u - u_h\|_{W^{1,p}} \leq C \left(\sum_{T\in\mathcal{T}_h} \eta_T^{p'}\right)^{1/p'}. \tag{8}$$

where,

$$\eta_T = \left\{ h_T \left|\nabla\left(|\nabla u_h|^{p-2}\nabla u_h\right) + f\right|_{0,p',T}^{p'} + \sum_{e\subset\partial T} h_e \left|\left[|\nabla u_h|^{p-2}\nabla u_h \cdot \vec{n}\right]_e\right|_{0,p',e}^{p'} \right\}^{1/p'}.$$

First, observe that,

$$\langle R, v\rangle = a(u_h - u, v), \quad \forall v \in W_0^{1,p}(\Omega). \tag{9}$$

by integrating by parts over each element $T \in \mathcal{T}_h$, we have,

$$\langle R, v\rangle = \sum_{T\in\mathcal{T}_h} \left\{ \int_T \left(-\mathrm{div}(|\nabla u_h|^{p-2}\nabla u_h) - f\right) v + \sum_{e\subset\partial T} \int_e \left[|\nabla u_h|^{p-2}\nabla u_h \cdot \vec{n}\right]_e v \right\}.$$

By Galerkin orthogonality, we have

$$\langle R, v_h\rangle = 0 \quad \text{for all } v_h \in V_h,$$

then, we obtain

$$\begin{aligned}
\langle R, v\rangle &= \langle R, v - \mathcal{I}_h v\rangle \\
&= \sum_{T\in\mathcal{T}_h} \left\{ \int_T \left(-\mathrm{div}(|\nabla u_h|^{p-2}\nabla u_h) - f\right)(v - \mathcal{I}_h v) + \sum_{e\subset\partial T} \int_e \left[|\nabla u_h|^{p-2}\nabla u_h \cdot \vec{n}\right]_e (v - \mathcal{I}_h v) \right\} \\
&= \langle A(u_h) - A(u), v - \mathcal{I}_h v\rangle .
\end{aligned}$$

and the interpolation operator $\mathcal{I}_h$, [1]

$$\|v-\mathcal{I}_h v\|_{m,p,T} \leq C h_T^{1-m} \sum_{T'\in S_T} \|v\|_{1,p,T'}, \quad \text{for } m=0,1.$$

we obtain,

$$\|R\|_{-1,p} \leq C\left(\sum_{T\in\mathcal{T}_h} \eta_T^{p'}\right)^{1/p'},$$

with,

$$\eta_T = \left\{ h_T \left|\nabla\left(|\nabla u_h|^{p-2}\nabla u_h\right)+f\right|_{0,p',T}^{p'} + \sum_{e\subset\partial T} h_e \left|\left[|\nabla u_h|^{p-2}\nabla u_h\cdot\vec{n}\right]_e\right|_{0,p',e}^{p'}\right\}^{1/p'}.$$

Now,

- for the case $p\geq 2$, we have,

$$\beta\|u-u_h\|_{1,p} \leq \sup_{v\in W_0^{1,p}} \frac{\langle A(u_h)-A(u),v\rangle}{\|v\|_{1,p}} = \sup_{v\in W_0^{1,p}} \frac{\langle R,v\rangle}{\|v\|_{1,p}} = \|R\|_{-1,p} \leq C\left(\sum_{T\in\mathcal{T}_h}\eta_T^{p'}\right)^{1/p'},$$

- For, $1<p\leq 2$, since we have $A(0)=0$, then,

$$\langle A(u),u\rangle \geq \alpha\|u\|_{1,p}^p.$$

and we have, $A(u)=f$ then,

$$\alpha\|u\|_{1,p}^p \leqslant \langle f,u\rangle \leq \|f\|_{W^{-1,p'}(\Omega)}\cdot\|u\|_{1,p}$$

i.e,

$$\|u\|_{1,p} \leq \left(\frac{1}{\alpha}\|f\|_{W^{-1,p'}(\Omega)}\right)^{1/(p-1)} \leq C\left(\|f\|_{L^{p'}(\Omega)}\right)^{1/(p-1)}$$

the same argument for for u_h, so,

$$\left(\|u\|_{1,p}+\|u_h\|_{1,p}\right)^{2-p} \leq C(\|f\|_{L^{p'}(\Omega)})^{\frac{2-p}{p-1}}$$

Now, from (3) we conclude that:

$$\|u_h-u\|_{1,p}^2 \lesssim \langle A(u_h)-A(u),u_h-u\rangle$$

or,

$$\|u_h-u\|_{1,p} \lesssim \sup_{v\in W_0^{1,p}} \frac{\langle A(u_h)-A(u),v-v_h\rangle}{\|v\|_{1,p}} = \|R\|_{-1,p} \leq C\left(\sum_{T\in\mathcal{T}_h}\eta_T^{p'}\right)^{1/p'},$$

[1] $sob(W^{m,p})+1-m = sob(W^{1,p})$.