Homework Assignment

to be submitted on 05/12/2025

December 5, 2025

Darcy Equations

Let Ω be a bounded connected open subset of $\mathbb{R}^2$, with polygonal boundary $\partial\Omega$ and outer unit normal vector n. We define

$$L_0^2(\partial\Omega) = \left\{ q \in L^2(\partial\Omega)\,;\, \int_{\partial\Omega} q(\sigma)\,d\sigma = 0 \right\}.$$

Let f be a given function in $L^2(\Omega)^2$ and g given in $L_0^2(\partial\Omega)$.

0.1 Preliminary Question

Let u be a function in $L^2(\Omega)^2$ such that

$$\forall q \in H^1(\Omega), \qquad \int_\Omega u \cdot \nabla q\,dx = \int_{\partial\Omega} g(\sigma)\,q(\sigma)\,d\sigma. \tag{1}$$

a. Show that, in the sense of distributions on Ω,

$$\operatorname{div} u = 0. \tag{2}$$

b. Assuming Green's formula holds, show that

$$u \cdot n = g \quad \text{on } \partial\Omega. \tag{3}$$

c. Conversely, still assuming Green's formula holds, show that if $u \in L^2(\Omega)^2$ satisfies (2) and (3), then u satisfies (1).

We consider the variational problem: find $u \in L^2(\Omega)^2$ and $p \in H^1(\Omega)$ such that

$$\forall v \in L^2(\Omega)^2, \qquad \int_\Omega u \cdot v\,dx + \int_\Omega v \cdot \nabla p\,dx = \int_\Omega f \cdot v\,dx, \tag{4}$$

$$\forall q \in H^1(\Omega), \qquad \int_\Omega u \cdot \nabla q\,dx = \int_{\partial\Omega} g(\sigma)\,q(\sigma)\,d\sigma. \tag{5}$$

II Study of the Exact Problem

2a. Write a system of partial differential equations with boundary conditions equivalent to (4)-(5). Prove the equivalence.

b. Show that the problem (4)-(5) remains unchanged if $H^1(\Omega)$ is replaced by $H^1(\Omega) \cap L^2_0(\Omega)$.

We endow the space $H^1(\Omega) \cap L^2_0(\Omega)$ with the norm $\|\nabla q\|_{L^2(\Omega)}$, which is equivalent to the usual H^1 norm. Let c_1 be the smallest constant such that:

$$\forall q \in H^1(\Omega) \cap L^2_0(\Omega), \qquad \|q\|_{L^2(\Omega)} \leq c_1 \|\nabla q\|_{L^2(\Omega)}.$$

Similarly, let c_2 be the smallest constant such that:

$$\forall q \in H^1(\Omega), \qquad \|q\|_{L^2(\partial\Omega)} \leq c_2 \|q\|_{H^1(\Omega)}.$$

We now define the spaces:

$$V = \left\{ v \in L^2(\Omega)^2 \,;\, \forall q \in H^1(\Omega) \cap L^2_0(\Omega),\ \int_\Omega v \cdot \nabla q \, dx = 0 \right\},$$

$$V(g) = \left\{ v \in L^2(\Omega)^2 \,;\, \forall q \in H^1(\Omega) \cap L^2_0(\Omega),\ \int_\Omega v \cdot \nabla q \, dx = \int_{\partial\Omega} g(\sigma) q(\sigma) \, d\sigma \right\},$$

$$V^\perp = \left\{ v \in L^2(\Omega)^2 \,;\, \forall \omega \in V,\ \int_\Omega v \cdot \omega \, dx = 0 \right\}.$$

3. Let $X = L^2(\Omega)^2$, $M = H^1(\Omega) \cap L^2_0(\Omega)$, and identify $L^2(\Omega)$ with its dual. Thus $X' = X$, and the duality pairing is the L^2 inner product.

   a. Using Question 1, give a characterization of V.

   b. Show that $V^0 = V^\perp$.

   c. For $q \in H^1(\Omega) \cap L^2_0(\Omega)$, show that

   $$\sup_{v \in L^2(\Omega)^2} \frac{\int_\Omega v \cdot \nabla q \, dx}{\|v\|_{L^2(\Omega)}} = \|\nabla q\|_{L^2(\Omega)}.$$

   d. Show that for every $f \in V^0$, there exists a unique $p \in H^1(\Omega) \cap L^2_0(\Omega)$ such that

   $$\nabla p = f.$$

   Give a characterization of $V^\perp$.

   e. Show that for all $g \in L^2_0(\partial\Omega)$, there exists a unique $\omega \in V^\perp$, denoted $\omega(g)$, such that $\omega(g) \in V(g)$ and

   $$\|\omega(g)\|_{L^2(\Omega)} \leq c_2 (c_1^2 + 1)^{1/2} \|g\|_{L^2(\partial\Omega)}.$$

f. Show that the problem (4)-(5) is equivalent to: find $u_0 \in V$ such that

$$\forall v \in V, \qquad \int_\Omega u_0 \cdot v \, dx = \int_\Omega f \cdot v \, dx,$$

and

$$u = u_0 + \omega(g).$$

g. Deduce that (4)-(5) admits a unique solution $u \in L^2(\Omega)^2$, $p \in H^1(\Omega) \cap L^2_0(\Omega)$, and that

$$\|u_0\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}, \qquad \|\nabla p\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)},$$
$$\|u\|^2_{L^2(\Omega)} \le \|f\|^2_{L^2(\Omega)} + c_2^2(c_1^2+1)\|g\|^2_{L^2(\partial\Omega)}.$$

(For this last inequality, expand $\|u_0 + \omega(g)\|^2_{L^2(\Omega)}$.)

III Discretization

Let $\mathcal{T}_h$ be a *shape-regular* family of triangulations of $\overline{\Omega}$ made of triangles T, with diameter $h_T \le h$. Define

$$X_h = \{v_h \in L^2(\Omega)^2\, ;\, v_h|_T \in \mathbb{P}_0^2 \text{ for all } T \in \mathcal{T}_h\},$$
$$M_h = \left\{ q_h \in C^0(\overline{\Omega})\, ;\, q_h|_T \in \mathbb{P}_1,\ \int_\Omega q_h(x)\, dx = 0 \right\}.$$

We discretize (4)-(5) by seeking $(u_h, p_h) \in X_h \times M_h$ satisfying:

$$\forall v_h \in X_h, \qquad \int_\Omega u_h \cdot v_h \, dx + \int_\Omega v_h \cdot \nabla p_h \, dx = \int_\Omega f \cdot v_h \, dx, \tag{6}$$

$$\forall q_h \in M_h, \qquad \int_\Omega u_h \cdot \nabla q_h \, dx = \int_{\partial\Omega} g(\sigma)\, q_h(\sigma)\, d\sigma. \tag{7}$$

4. Show that (6)-(7) admits a unique solution $u_h \in X_h$, $p_h \in M_h$.

5. a. Define the spaces V_h, $V_h^\perp$, and $V_h(g)$. Give a characterization of $V_h^\perp$.

   b. For $q_h \in M_h$, show that

$$\sup_{v_h \in X_h} \frac{\int_\Omega v_h \cdot \nabla q_h \, dx}{\|v_h\|_{L^2(\Omega)}} = \|\nabla q_h\|_{L^2(\Omega)}.$$

   c. Show that for each $g \in L^2_0(\partial\Omega)$, there exists a unique $\omega_h \in V_h^\perp$, denoted $\omega_h(g)$, such that $\omega_h(g) \in V_h(g)$ and

$$\|\omega_h(g)\|_{L^2(\Omega)} \le c_2(c_1^2+1)^{1/2}\|g\|_{L^2(\partial\Omega)}.$$

d. By decomposing $u_h = u_{h,0} + \omega_h(g)$, show that the solution (u_h, p_h) of (6)-(7) satisfies estimates analogous to those from Question 3g.

6. Let (u, p) be the solution of (4)-(5). Define $P_h(u) \in X_h$ by

$$\forall T \in \mathcal{T}_h, \qquad P_h(u)|_T = \frac{1}{|T|}\int_T u(x)\,dx,$$

and let $R_h : H^1(\Omega) \cap L^2_0(\Omega) \to M_h$ be the regularization operator described on p.92 of the course.

a. Show that $P_h(u) \in V_h(g)$.

b. Show that

$$\|u_h - P_h(u)\|_{L^2(\Omega)} \leq \|u - P_h(u)\|_{L^2(\Omega)} + \|\nabla(p - R_h(p))\|_{L^2(\Omega)},$$

$$\|\nabla(p_h - R_h(p))\|_{L^2(\Omega)} \leq \|\nabla(p - R_h(p))\|_{L^2(\Omega)}.$$

c. Assuming $(u, p) \in H^1(\Omega)^2 \times H^2(\Omega)$, give an estimate for the errors

$$\|u_h - u\|_{L^2(\Omega)} \qquad \text{and} \qquad \|\nabla(p_h - p)\|_{L^2(\Omega)}.$$

7. We now seek a better estimate for $\|p_h - p\|_{L^2(\Omega)}$. Using duality, write

$$\|p_h - p\|_{L^2(\Omega)} = \sup_{q \in L^2_0(\Omega)} \frac{\int_\Omega (p_h - p)q\,dx}{\|q\|_{L^2(\Omega)}}.$$

a. Define $\varphi \in H^1(\Omega) \cap L^2_0(\Omega)$ as the solution of

$$-\Delta\varphi = q \quad \text{in } \Omega, \qquad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Show that for any $\varphi_h \in M_h$,

$$\int_\Omega (p_h - p)q\,dx = \int_\Omega \nabla(p_h - p) \cdot \nabla(\varphi - \varphi_h)\,dx.$$

b. Deduce that if Ω is convex, then

$$\|p_h - p\|_{L^2(\Omega)} \leq Ch^2|p|_{H^2(\Omega)}.$$