2024-2025

Master :AN/AF　　　　Kasdi Merbah Univ.

Exam.

1 QMC. (8 pts)

Find the correct answer and **justify your choice**.

Q1) Find the variational problem (PV) for the following classical problem :

$$-u'' = f, \quad u(0) = u(1) = 0. \tag{PC}$$

☐ Find $u \in H_0^1(0,1)$ such that $\int_0^1 u'v'\,dx = \langle f, v\rangle\,, \forall v \in H_0^1(0,1)$.
☐ Find $u \in H_0^1(0,1)$ such that $\int_0^1 u'v'\,dx = \langle f, v\rangle\,, \forall v \in H^1(0,1)$.
☐ Find $u \in H^1(0,1)$ such that $\int_0^1 u'v'\,dx = \langle f, v\rangle\,, \forall v \in H^1(0,1)$.

Q2) Let $b \in W^{1,\infty}(0,1)$, $c \in L^\infty(0,1)$ and $f \in L^2(0,1)$. We consider the following boundary value problem :

$$-u'' + b(x)u' + c(x)u = f(x), \quad 0 < x < 1, \tag{1}$$
$$u(0) = u(1) = 0, \tag{2}$$

Then, the corresponding variational problem is well posed if :
☐ $c(x) + b(x) > 0$, for $x \in (0,1)$.
☐ $c(x) - \frac{1}{2}b'(x) \leq 0$, for $x \in (0,1)$.
☐ $c(x) - \frac{1}{2}b'(x) \geq 0$, for $x \in (0,1)$.

Q3) Find the maximal regularity of the solution for the following variational problem :

$$\begin{cases} \text{Find } u \in V = \{v \in H^1(0,1); v(0) = 0\}, \text{ such that} \\ \displaystyle\int_0^1 u'v'dx = \int_0^{1/2} v dx, \quad \forall v \in V \end{cases} \tag{3}$$

☐ $u \in H^1(0,1) \cap V$.
☐ $u \in H^2(0,1) \cap V$.
☐ $u \in H^3(0,1) \cap V$.

Q4) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0,1]$ with maximum width h, and let $\mathcal{I}_h : H^2(0,1) \to V_h$ be the Lagrange interpolation operator onto V_h. Then, $\exists C > 0$ such that
☐ $\|v - \mathcal{I}_h(v)\|_0 \leq C\,h|v|_2, \quad \forall v \in H^2(0,1)$.
☐ $\|v - \mathcal{I}_h(v)\|_1 \leq C\,h|v|_2, \quad \forall v \in H^2(0,1)$.
☐ $\|v - \mathcal{I}_h(v)\|_2 \leq C\,h|v|_2, \quad \forall v \in H^2(0,1)$.

2 Problem (12 pts)

Let's consider the mixed Poisson equation in one dimension. Start with

$$-u'' = f, \quad u(0) = 0 = u(1) \tag{4}$$

and introduce $\sigma = -u'$ to get the system

$$\begin{cases} \sigma + u' = 0 \\ \sigma' = f \end{cases} \tag{5}$$

1. Write the system (5) in a mixted form [1] :

$$\begin{cases} \text{find } (\sigma, u) \in H^1(0,1) \times L^2(0,1) \text{ such that} \\ a(\sigma, \tau) + b(\tau, u) = 0, \quad \forall \tau \in H^1(0,1) \\ b(\sigma, v) = F(v), \quad \forall v \in L^2(0,1). \end{cases} \tag{6}$$

2. Show that the mixed problem (6) is well posed.

3. By using the Fortin trick, show that the discretisation $\mathbb{P}_1 \times \mathbb{P}_0$ is well posed.

3 Bonus (2 pts)

Write a Freefem++ code that solves the discrete counterpart of the mixed problem (6).

1. precise the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and the linear form $F(\cdot)$.

Answers.

A. 1. Take $v \in H_0^1(0,1)$ and multiply the equation $-u'' = f$ by v and integrate by part we obtain :

$$\int_0^1 u'v'\,dx + \underbrace{[u'v]_0^1}_{=0} = \langle f, v\rangle$$

Hence, the variational problem is :

$$\begin{cases} \text{Find } u \in H_0^1(0,1) \text{ such that} \\ \displaystyle\int_0^1 u'v'\;dx = \langle f, v\rangle\,, \forall v \in H_0^1(0,1). \end{cases} \tag{7}$$

A. 2. We need to show that there exists a constant $\alpha > 0$ such that :

$$a(v,v) \geq \alpha \|v\|^2_{H_0^1(0,1)}.$$

Let's evaluate $a(v,v)$:

$$a(v,v) = \int_0^1 v'^2\,dx + \int_0^1 b(x)v'v\,dx + \int_0^1 c(x)v^2\,dx.$$

Using integration by parts on the middle term $\int_0^1 b(x)v'v\,dx$, we get :

$$\int_0^1 b(x)v'v\,dx = \frac{1}{2}\int_0^1 b(x)\frac{d}{dx}v^2\,dx = -\frac{1}{2}\int_0^1 b'(x)v^2\,dx.$$

So, the bilinear form becomes :

$$a(v,v) = \int_0^1 v'^2\,dx + \int_0^1 \left(c(x) - \frac{1}{2}b'(x)\right) v^2\,dx.$$

Hence if the condition $c(x) - \frac{1}{2}b'(x) \geq 0$, is satisfied then we have :

$$a(v,v) \geq \int_0^1 v'^2\,dx,$$

which shows coercivity, since $\|v\|^2_{H_0^1(0,1)} = \|v'\|^2_{L^2(0,1)}$.

A. 3. The variational problem

$$\begin{cases} \text{Find } u \in V = \{v \in H^1(0,1); v(0) = 0\}, \text{ such that} \\ \displaystyle\int_0^1 u'v'dx = \int_0^{1/2} vdx, \quad \forall v \in V \end{cases} \tag{8}$$

is at least formally equivalent to the following classical boundary values problem :

$$\begin{aligned} -u'' =& H_{1/2}(x) \\ u(0) =& 0 \end{aligned}$$

where,

$$H_{1/2}(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1/2 \\ 0 & \text{if } 1/2 < x \le 1 \end{cases} \tag{9}$$

and thereore, the exact solution is

$$u(x) = \begin{cases} \dfrac{x(1-x)}{2} & \text{if } 0 \le x \le 1/2 \\ 1/8 & \text{if } 1/2 < x \le 1 \end{cases} \tag{10}$$

with

$$u'(x) = \begin{cases} \dfrac{1}{2} - x & \text{if } 0 \le x \le 1/2 \\ 0 & \text{if } 1/2 < x \le 1 \end{cases} \tag{11}$$

and

$$u''(x) = \begin{cases} -1 & \text{if } 0 \le x \le 1/2 \\ \ \ 0 & \text{if } 1/2 < x \le 1 \end{cases} \tag{12}$$

So, $u \in C^1[0,1] \cap V$, $u'' \in L^2(0,1)$ and $u''' \notin L^2(0,1)$.
Hence,

$$u \in H^2(0,1) \cap V.$$

A. 4. The interpolation theory claims that $\exists C > 0$ such that :

$$\|v - \mathcal{I}_h(v)\|_\ell \le C\, h^{m-\ell} |v|_m, \quad \forall v \in H^m(0,1).$$

Hence for $m = 2$ and $\ell = 1$ we have

$$\|v - \mathcal{I}_h(v)\|_1 \le C\, h |v|_2, \quad \forall v \in H^2(0,1).$$

The mixed problem

1. Testing the equations with $(\tau, v) \in V \times Q = H^1(0,1) \times L^2(0,1)$, we get

$$\int_\Omega \sigma\tau \mathrm{d}x + \int_\Omega u'\tau \mathrm{d}x = 0$$
$$\int_\Omega \sigma' v \ \mathrm{d}x = \int_\Omega f v \ \mathrm{d}x$$

Let's integrate by parts to remove the derivative from u onto τ, and negate :

$$\int_\Omega \sigma\tau \mathrm{d}x - \int_\Omega u\tau' \mathrm{d}x + \int_{\partial\Omega} u\tau \ \mathrm{d}s = 0$$
$$-\int_\Omega \sigma' v \ \mathrm{d}x = -\int_\Omega f v \ \mathrm{d}x$$

We thus have :

$$\begin{cases} \text{find } (\sigma, u) \in H^1(\Omega) \times L^2(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \sigma\tau \mathrm{d}x - \int_\Omega u\tau' \mathrm{d}x = 0 \\ \displaystyle\qquad -\int_\Omega \sigma' v \ \mathrm{d}x = -\int_\Omega f v \ \mathrm{d}x \end{cases} \tag{13}$$

for all $(\tau, v) \in V \times Q$.

$$a(\sigma, \tau) = \int_\Omega \sigma\tau \, dx, \quad b(\tau, v) = -\int_\Omega v\tau' \, dx, \qquad F(v) = -\int_\Omega f v \, dx$$

2. Let's think about well-posedness. Is

$$a(\sigma, \tau) = \int_\Omega \sigma\tau \mathrm{d}x = (\sigma, \tau)_{L^2(\Omega)}$$

coercive over the kernel

$$\ker b := \left\{ \tau \in H^1(\Omega) : \int_\Omega \tau' v \ \mathrm{d}x = 0 \text{ for all } v \in L^2(\Omega) \right\}?$$

Since $\tau' \in L^2(\Omega)$, choosing $v = \tau'$ as test function yields that

$$\tau \in \ker b \iff \tau' = 0 \iff \tau = Cte$$

Then, if $\tau \in \ker b$,
$$a(\tau, \tau) = \|\tau\|_{L^2}^2 = \|\tau\|_{H^1}^2$$
Hence, $a(\cdot, \cdot)$ is coercive on $\ker b$.

For the inf-sup condition, we require that there exists γ such that

$$0 < \gamma \leq \inf_{\substack{v \in L^2(\Omega) \\ v \neq 0}} \sup_{\substack{\tau \in H^1(\Omega) \\ \tau \neq 0}} \frac{\int_\Omega \tau' v \, \mathrm{d}x}{\|\tau\|_{H^1(\Omega)} \|v\|_{L^2(\Omega)}}$$

For a given $v \in L^2(\Omega)$, choose

$$\tau(x) = \int_0^x v(x) \mathrm{d}x$$

so that $\tau' = v$ and $\tau(0) = 0$.
We know that for such a function

$$\|\tau\|_{L^2(\Omega)} \leq c \, \|\tau'\|_{L^2(\Omega)} = c\|v\|_{L^2(\Omega)}$$

and hence

$$\|\tau\|_{H^1(\Omega)} \leq c\|v\|_{L^2(\Omega)}$$

for some (different) c.
With this choice, for any $v \in L^2(\Omega)$,

$$\sup_{\substack{\tau \in V \\ \tau \neq 0}} \frac{\int_\Omega \tau' v \, \mathrm{d}x}{\|\tau\|_{H^1(\Omega)} \|v\|_{L^2(\Omega)}} \geq \frac{\|v\|^2_{L^2(\Omega)}}{c\|v\|^2_{L^2(\Omega)}} = \frac{1}{c} > 0$$

so the inf-sup condition holds. Applying Brezzi's theorem, we conclude that the mixed formulation is well-posed.

3. We need to show that :

i) For each $\tau \in V = H^1$ there is an element $\Pi_h(\tau) \in V_h = \mathbb{P}_1$ such that :

$$b(\tau, v_h) = b(\Pi_h(\tau), v_h) \quad \forall v_h \in Q_h = \mathbb{P}_0$$

ii) There exists a constant $C > 0$ such that :

$$\|\Pi_h(\tau)\|_V \leq C\|\tau\|_V.$$

To show the condition i), we need to find $\Pi_h \tau$ such that :

$$\int_0^1 (\tau - \Pi_h(\tau))' v_h = 0$$

this can be done if

$$\int_{x_i}^{x_{i+1}} (\tau - \Pi_h(\tau))' \times 1 = 0$$

So one can take $\Pi_h \tau = \mathcal{I}_h(\tau)$ the Lagrange interpolant, because

$$\mathcal{I}_h(\tau)(x_i) = \tau(x_i), \quad \forall i$$

The second condition ii) is a direct consequence of the stability of Lagrange interpolant.