FEM for Nonlinear Problems
Homework 1

Ex.1 Consider the 1d Bratu problem:

$$\begin{cases} u'' + \lambda e^u = 0 & \text{in } \Omega =]0,1[\\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

i) Check that the solutions of (1) are given by:

$$u(x) = 2\ln\left(\frac{\cosh(\alpha)}{\cosh(\alpha(1-2x))}\right) \tag{2}$$

where α satisfies the transcendental equation

$$\cosh(\alpha) = \frac{4}{\sqrt{2\lambda}}\alpha \tag{3}$$

ii) Show that for[1] :

a) $\lambda = 1$, problem (1) admits two solutions.

b) $\lambda = \lambda^* = 3.513830719$, problem (1) admits a unique solution.

c) $\lambda \notin [0, \lambda^*]$ there are no solutions of (1).

iii) Given an initial guess u_0, state the linearised system of equations that must be solved for the update δu in a Newton-Kantorovich iteration, in weak form.

Ex.2 Consider the problem:

$$u = \underset{v \in H^1(\Omega)}{\operatorname{argmin}} J(v) \tag{4}$$

where

$$J(v) = \frac{1}{2}\int_\Omega \gamma \nabla v \cdot \nabla v + \frac{1}{2}\left(v^2 - 1\right)^2 \, \mathrm{d}x \tag{5}$$

i) State the Euler-Lagrange equation that characterises stationary points u of this functional.

ii) Given an initial guess u_0, state the linearised system of equations that must be solved for the update δu in a Newton-Kantorovich iteration, in weak form.

[1]You may use the matlab function fzero

Ex.3 Consider the stationary incompressible isothermal Newtonian Navier-Stokes equations, given in strong form by

$$\begin{aligned} -\nabla^2 u + (u \cdot \nabla)u + \nabla p &= f \text{ in } \Omega, \\ \nabla \cdot u &= 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{6}$$

i) Write this system of equations in weak form as a nonlinear variational equation.

ii) Write the linearised system of equations that must be solved for the update (d_u, d_p) in a Newton-Kantorovich iteration, in weak and strong form.

Ex.4 Let V and Y be Banach spaces. Consider the Newton-Kantorovich iteration applied to

i) $F : V \to V^*$;

ii) $(G \circ F) : V \to Y$, where $G : V^* \to Y$ is linear and bijective.

iii) Prove that the Newton-Kantorovich iteration yields the same sequence of iterates in both cases when initialised from the same initial guess $u_0 \in V$.

Ex.5 (Difficult) Let $Z : V \to V$ be an invertible operator. Suppose that it is a symmetry of a residual $F : V \to V^*$, i.e. it satisfies

$$ZRF(u) = RF(Zu) \tag{7}$$

for all $u \in V$, where $R : V^* \to V$ is the Riesz map. (Imagine, for example, that Z were a reflection; this property asserts that the reflection of the residual is the residual of the reflection.)

Prove that if the Newton-Kantorivich iteration is initialised from an initial guess u_0 that satisfies $Zu_0 = u_0$, then all subsequent iterates u_k will also satisfy $Zu_k = u_k$, so long as the iterates are defined.

Remark: this means that one should be careful when solving nonlinear problems with symmetries. In general, such problems will support both symmetric and nonsymmetric solutions. If the initial guess is chosen to be symmetric, then only symmetric solutions will be found by the iteration if the linear subproblem is solved exactly.