A posteriori error analysis for the obstacle problem

1 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^2$ with boundary $\partial\Omega$. Let $f \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega)\cap C(\bar{\Omega})$. We introduce :

$$\begin{aligned} V :=& H_0^1(\Omega) := \{v \in H^1(\Omega) \quad | \quad v = 0 \text{ on } \partial\Omega\} \\ \mathcal{K} :=& \{v \in V; \quad v \geq \psi \text{ a.e. in } \Omega\} \\ Q :=& H^{-1}(\Omega) \end{aligned}$$

We are interested in the a posteriori analysis of the following variational inequality :

$$\begin{cases} \text{Find } u \in \mathcal{K} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v-u)dx \geq \int_\Omega f(v-u)dx, \quad \forall v \in \mathcal{K} \end{cases} \tag{1}$$

<u>We recall that</u> (see the MEF2 course) :

- ☞ Stampacchia's theorem guarantees the existence of a unique solution u to (1).
- ☞ If $\partial\Omega$ is sufficiently regular ($C^{1,1}$), then $u \in H^2(\Omega) \cap H_0^1(\Omega)$.
- ☞ u satisfies the complementarity system :

$$\begin{cases} -\Delta u - f \geq 0, & \text{in } \Omega \\ u - \psi \geq 0, & \text{in } \Omega \\ (u-\psi, -\Delta u - f) = 0, & \text{in } \Omega \end{cases} \tag{2}$$

- ☞ *A priori* error estimation. If $u \in H^2(\Omega) \cap \mathcal{K}$ is the solution of (1) and $u_h \in \mathcal{K}_h$ (with $V_h = V_h^1$) is the solution of the discrete problem, then we have :

$$|u-u_h|_{1,\Omega} = \|\nabla(u-u_h)\|_{L^2(\Omega)} \leq C\ h \left(|u|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)} + |\psi|_{H^2(\Omega)}\right) \tag{3}$$

Remarks :

1. The estimation (3) requires $H^2(\Omega)$ regularity.
2. Practically, it is difficult to verify $\mathcal{K}_h \subset \mathcal{K}$.

2 A Mixed Variational Formulation

We introduce the set :

$$\Lambda := \{\mu \in Q \quad | \quad \langle \mu, v\rangle \geq 0, \quad \forall v \in V, \quad v \geq 0 \text{ a.e. in } \Omega\}$$

Let $\mathcal{H} = V \times \Lambda$. We define the bilinear form $\mathcal{A} : \mathcal{H} \to \mathbb{R}$ and the linear form $\mathcal{L} : V \to \mathbb{R}$ by :

$$\mathcal{A}((v, \xi); (w, \mu)) = (\nabla v, \nabla w) - \langle \xi, w\rangle - \langle \mu, v\rangle$$
$$\mathcal{L}(w, \mu) = (f, w) - \langle \psi, \mu\rangle$$

1. Verify that the problem (1) is equivalent to the following variational problem :

$$\begin{cases} \text{Find } (u, \lambda) \in \mathcal{H} \quad \text{such that} \\ \mathcal{A}((u, \lambda); (w,\ \mu - \lambda)) \leq \mathcal{L}(w, \mu - \lambda), \quad \forall (w, \mu) \in \mathcal{H} \end{cases} \tag{4}$$

2. Show that there exists $\beta > 0$ such that :

$$\inf_{\xi \in Q} \sup_{v \in V} \frac{\langle \xi, v\rangle}{\|v\|_V \|\xi\|_Q} \geq \beta \tag{5}$$

3. Show that for all $(v, \xi) \in \mathcal{H}$, there exists $w \in V$ such that :

$$\mathcal{A}((v, \xi); (w, -\xi)) \gtrsim (\|v\|_1 + \|\xi\|_{-1})^2 \tag{6}$$
$$\|w\|_1 \lesssim \|v\|_1 + \|\xi\|_{-1} \tag{7}$$

4. Deduce that :

$$\exists \beta > 0, \quad \text{such that} \quad \sup_{(w,\mu) \in \mathcal{H}} \frac{\mathcal{A}((v, \xi); (w, \mu))}{\|(w, \mu)\|_{\mathcal{H}}} \geq \beta \|(v, \xi)\|_{\mathcal{H}}$$

3 Discretization of the Mixed Problem

Let $\mathcal{T}_h$ be a regular triangulation of Ω, and $\mathcal{E}_h$ the set of interior edges.
We consider two finite-dimensional spaces V_h and Q_h :

$$V_h \subset H^1_0(\Omega) \quad Q_h \subset H^{-1}(\Omega)$$

where the functions of Q_h are piecewise polynomial. Additionally, we define :

$$\Lambda_h = \{\mu_h \in Q_h : \mu_h \geq 0 \quad \text{in } \Omega\} \subset \Lambda$$

and consider the following discrete problem :

$$\begin{cases} \text{Find } (u_h, \lambda_h) \in V_h \times \Lambda_h \text{ such that} \\ \mathcal{A}((u_h, \lambda_h); (v_h, \mu_h - \lambda_h)) \leq \mathcal{L}(v_h, \mu_h - \lambda_h) \quad \forall (v_h, \mu_h) \in V_h \times \Lambda_h \quad \forall \mu_h \in \Lambda_h \end{cases} \tag{8}$$

1. Show that if V_h and Q_h are such that :

$$\sup_{v_h\in V_h}\frac{\langle v_h,\xi_h\rangle}{\|v_h\|_1}\gtrsim\|\xi_h\|_{-1},\quad\forall\xi_h\in Q_h \tag{9}$$

Then,

a. For all $(v_h,\xi_h)\in V_h\times\Lambda_h$, there exists $w_h\in V_h$ such that :

$$\mathcal{A}((v_h,\xi_h);(w_h,-\xi_h))\gtrsim(\|v_h\|_1+\|\xi_h\|_{-1})^2 \tag{10}$$

$$\|w_h\|_1\lesssim\|v_h\|_1+\|\xi_h\|_{-1} \tag{11}$$

b. Deduce that :

$$\exists\tilde{\beta}>0,\quad\text{such that}\quad\sup_{(w_h,\mu_h)\in\mathcal{H}_h}\frac{\mathcal{A}((v_h,\xi_h);(w_h,\mu_h))}{\|(w_h,\mu_h)\|_{\mathcal{H}}}\geq\tilde{\beta}\|(v_h,\xi_h)\|_{\mathcal{H}} \tag{12}$$

2. Show that the mixed formulation (8) has a unique solution $(u_h,\lambda_h)\in\mathcal{H}_h$.

3. Approximation Property. Assume that $(u,\lambda)\in\mathcal{H}$ is the solution of (4) and $(u_h,\lambda_h)\in\mathcal{H}_h$ is the solution of (8). Then :

$$\|(u,\lambda)-(u_h,\lambda_h)\|_{\mathcal{H}}=\inf_{(v_h,\xi_h)\in\mathcal{H}_h}\|(u,\lambda)-(v_h,\xi_h)\|_{\mathcal{H}}$$

4 A Posteriori Error Estimation

First, we define the local indicators :

$$\eta_T^2=h_T^2\|\Delta u_h+\lambda_h+f\|_{0,T}^2,$$
$$\eta_e^2=h_e\|[\![\nabla u_h\cdot n]\!]\|_{0,e}^2.$$

We also define :

$$\eta^2=\sum_T\eta_T^2+\sum_e\eta_e^2,$$
$$S_m=\|\psi-u_h\|_1+\sqrt{\langle(\psi-u_h)_+,\lambda_h\rangle},$$

where $w_+=\max\{w,0\}$ denotes the positive part of w.

4.1 Reliability of the Indicator

In this subsection, we aim to prove the following estimate :

$$\|u-u_h\|_1+\|\lambda-\lambda_h\|_{-1}\lesssim\eta+S_m. \tag{$***$}$$

1. Show that there exists $w\in V$ such that :

$$\mathcal{A}((u-u_h,\lambda-\lambda_h);(w,\lambda_h-\lambda))\gtrsim(\|u-u_h\|_1+\|\lambda-\lambda_h\|_{-1})^2, \tag{13}$$

$$\|w\|_1\lesssim\|u-u_h\|_1+\|\lambda-\lambda_h\|_{-1}. \tag{14}$$

2. Let w_h be the Clément interpolation of w. Show that :

$$0 \leq \mathcal{L}(-w_h, 0) - \mathcal{A}((u_h, \lambda_h); (-w_h, 0)).$$

3. Show that :

$$(\|u - u_h\|_1 + \|\lambda - \lambda_h\|_{-1})^2 \lesssim \mathcal{L}(w - w_h, \lambda_h - \lambda) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)). \quad (15)$$

4. Show that :

$$\langle u_h - \psi, \lambda_h - \lambda \rangle \leq \|(\psi - u_h)_+\|_1 \|\lambda - \lambda_h\|_{-1} + \langle (\psi - u_h)_+, \lambda_h \rangle . \quad (16)$$

5. Deduce the estimate $(***)$.

4.2 Optimality of the Indicator

Let $f_h \in Q_h$ be the projection of f with respect to the L_2 inner product, and define :

$$osc_T(f) = h_T \|f - f_h\|_{0,T},$$
$$osc(f)^2 = \sum_T osc_T(f)^2.$$

1. Show that for any $v_h \in V_h$ and $\mu_h \in Q_h$, we have :

$$h_T \|\Delta v_h + \mu_h + f\|_{0,T}^2 \lesssim \|u - v_h\|_{1,T} + \|\lambda - \mu_h\|_{-1,T} + osc_T(f), \quad (17)$$
$$h_e^{1/2} \|\nabla v_h \cdot n\|_{0,e} \lesssim \|u - v_h\|_{1,\omega(e)} + \sum_{T \in \omega(e)} (\|\lambda - \mu_h\|_{-1,T} + osc_T(f)) . \quad (18)$$

2. Deduce that :

$$\eta_T \lesssim \|u - u_h\|_{1,T} + \|\lambda - \lambda_h\|_{-1,T} + osc_T(f), \quad (19)$$
$$\eta_e \lesssim \|u - u_h\|_{1,\omega(e)} + \sum_{T \in \omega(e)} (\|\lambda - \lambda_h\|_{-1,T} + osc_T(f)) , \quad (20)$$
$$\eta \lesssim \|u - u_h\|_1 + \|\lambda - \lambda_h\|_{-1} + osc(f). \quad (21)$$