Homework 1: A Posteriori Error Analysis by Duality

1 A general diffusion operators with variable coefficients.

Let Ω be a bounded open set of $\mathbb{R}^d$ with boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$. We consider the following problem:

$$\begin{cases} Lu := -\nabla \cdot \{a\nabla u\} + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ n \cdot \{a\nabla u\} = g & \text{on } \Gamma_N. \end{cases} \tag{1}$$

We define the residuals, $R(u_h)_{|K}$ and $r(u_h)_{|\Gamma}$, by

$$R(u_h)_{|K} := f + \nabla \cdot \{a\nabla u_h\} - cu_h \tag{2}$$

$$r(u_h)_{|\Gamma} := \begin{cases} \frac{1}{2} n \cdot [a\nabla u_h], & \text{if } \Gamma \subset \partial K \backslash \partial\Omega \\ 0, & \text{if } \Gamma \subset \Gamma_D \\ n \cdot \{a\nabla u_h\} - g, & \text{if } \Gamma \subset \Gamma_N \end{cases} \tag{3}$$

The error e is to be estimated in the L^2 norm using a posteriori error estimates by duality method.

1. By considering the following adjoint problem :

$$-\nabla \cdot \{a\nabla z\} = \|e\|^{-1} e \qquad \text{in } \Omega, \tag{4}$$

$$z = 0 \qquad \text{on } \partial\Omega \tag{5}$$

   show that:

$$\|e\| \leq \eta_{L^2}(u_h) := c_I c_s \left(\sum_{K \in \mathbb{T}_h} h_K^4 \rho_K^2 \right)^{1/2}, \quad c_s := \left\| \nabla^2 z \right\|$$

   and the 'weighted' L^2-error estimate

$$\|e\| \leq \eta_{L^2}^{\omega}(u_h) := c_I \sum_{K \in \mathbb{T}_h} h_K^2 \rho_K \omega_K, \quad \omega_K := \left\| \nabla^2 z \right\|_K$$

   with the cell residuals $\rho_K := \|R(u_h)\|_K + h_K^{-1/2} \|r(u_h)\|_{\partial K}$ as defined above. Both error estimators are evaluated by replacing the second derivatives of the dual solution z by second-order difference quotients of an approximation $z_h \in V_h$,

$$\omega_K \approx \tilde{\omega}_K := \left\| \nabla_h^2 z_h \right\|_K, \quad c_s \approx \tilde{c}_s := \left(\sum_{K \in \mathbb{T}_h} \left\| \nabla_h^2 z_h \right\|_K^2 \right)^{1/2}$$

   The interpolation constant is set to $c_I = 0.2$. The functional $J(\cdot)$ is evaluated by replacing the unknown solution u by a patch-wise higher-order interpolation $I_h^{(2)} u_h$ of the computed approximation u_h, that is, $e \approx I_h^{(2)} u_h - u_h$.

2. Numerical test: We choose the square domain $\Omega = (-1, 1) \times (-1, 1)$ and the nonconstant coefficient function $a(x) = 0.1 + e^{3(x_1 + x_2)}$ with right-hand side $f \equiv 0.1$. A reference solution is generated by a computation on a very fine mesh. The meshes are refined according to the 'error-balancing strategy' by balancing (as well as possible) the cell-wise indicators:

$$\eta_K := h_K^4 \rho_K^2 \quad \text{or} \quad \eta_K := h_K^2 \rho_K \tilde{\omega}_K$$

   Show error plots and meshes obtained by the two different strategies.