<u>Exercises and Problems</u>

1 Exercises

Ex.1. Consider the following triplet $(K, \mathcal{P}, \mathcal{N})$

i) $K = [0, 1]$.

ii) $\mathcal{P}$ is the linear polynomials on K, i.e $\mathbb{P}_1$.

iii) $\mathcal{N}$ is :

$$N_1(p) = p(0.5), \quad N_2(p) = \int_0^1 p(x)\,dx.$$

Is this finite element unisolvent? Explain your answer

Ex.2. a) Obtain the nodal basis function $\phi_1(x)$ for the finite element $(K = [0, 1], \mathcal{P}_2, \mathcal{N})$, with $\mathcal{N} = (N_1, N_2, N_3)$ given by

$$\begin{aligned} N_1(f) &= f(0), \\ N_2(f) &= f(1), \\ N_3(f) &= \int_0^1 f dx. \end{aligned}$$

b) What is the global continuity of finite element spaces constructed from the finite element described above ? Explain your answer.

Ex.3. Consider the finite element $(K, \mathcal{P}, \mathcal{N})$ where

- K is a non-degenerate triangle.
- $\mathcal{P}$ is the space of polynomials of degree 2 or less.
- $\mathcal{N} = (N_1, N_2, N_3, N_4, N_5, N_6)$ with

$$\begin{aligned} N_i(v) &= v\left(z_i\right), \qquad i = 1, 2, 3, \\ N_4(v) &= v\left(\frac{z_1 + z_2}{2}\right), \\ N_5(v) &= v\left(\frac{z_1 + z_3}{2}\right), \\ N_6(v) &= v\left(\frac{z_2 + z_3}{2}\right), \end{aligned}$$

where z_1, z_2 and z_3 are the vertices of K.

Show that $\mathcal{N}$ determines $\mathcal{P}$.

Ex.4. Now consider the finite element $(K, \hat{\mathcal{P}}, \hat{\mathcal{N}})$ where

- K is a non-degenerate triangle (with boundary ∂K).
- $\hat{\mathcal{P}}$ is the space spanned by polynomials of degree 2 or less, plus the cubic "bubble" function $B(x)$ satisfying $B(x) = 0$ for all $x \in \partial K$, and $\int_K B(x)\mathrm{d}x = 1$.
- $\hat{\mathcal{N}} = (N_1, N_2, N_3, N_4, N_5, N_6, N_7)$ with N_i as above for $i = 1, \ldots, 6$, and
- $$N_7(v) = v\left(\frac{z_1 + z_2 + z_3}{3}\right).$$

Show that $\hat{N}$ determines $\hat{P}$.

Ex.5. What is the choice of the geometric decomposition (allocation of nodal variables to cell and vertex entities) that leads to the maximum possible global continuity of finite element spaces defined on the interval $[0, L]$ constructed from the following one-dimensional elements (K, P, N). Justify your answer.

a) $K = [a, b], P$ is linear polynomials, $N = (N_1, N_2)$ where

$$N_1[u] = u((a + b)/2), \quad N_2[u] = u'((a + b)/2).$$

b) $K = [a, b], P$ is quadratic polynomials, $N = (N_1, N_2, N_3)$ where

$$N_1[u] = u(a), \quad N_2[u] = u(b), \quad N_3[u] = \int_a^b u \, \mathrm{d}x.$$

c) $K = [a, b], P$ is quadratic polynomials, $N = (N_1, N_2, N_3)$ where

$$N_1[u] = u'(a), \quad N_2[u] = u'(b), \quad N_3[u] = u((a + b)/2).$$

Ex.6. Consider the interval $[a, b]$, with points :

$$a = x_0, x_1, x_2, \ldots, x_{n-1}, x_n = b.$$

Let $\mathcal{T}$ be a subdivision (i.e. a 1D mesh) of the interval $[a, b]$ into subintervals

$$I_k = [x_k, x_{k+1}], k = 0, \ldots, N - 1.$$

Consider the following three elements.

(a) (K, P, N) where $K = I_k$, P are polynomials of degree $\leqslant 3$, and $N = (N_1, N_2, N_3, N_4)$ with:

$$\begin{aligned} N_1[u] &= u(x_k), & N_2[u] &= u(x_{k+1}), \\ N_3[u] &= \int_{x_k}^{x_{k+1}} u dx, & N_4[u] &= u'((x_{k+1} + x_k)/2). \end{aligned}$$

(b) (K, P, N) where $K = I_k$, P are polynomials of degree $\leqslant 3$, and $N = (N_1, N_2, N_3, N_4)$ with:

$$\begin{aligned} N_1[u] &= u(x_k), & N_2[u] &= u(x_{k+1}), \\ N_3[u] &= u'(x_k), & N_4[u] &= u'(x_{k+1}). \end{aligned}$$

(c) (K, P, N) where $K = I_k$, P are polynomials of degree $\leqslant 3$, and $N = (N_1, N_2, N_3, N_4)$ with:

$$N_1[u] = u\left(\left(x_{k+1} + x_k\right)/2\right), \ \ N_2[u] = u'\left(\left(x_{k+1} + x_k\right)/2\right),$$
$$N_3[u] = u''\left(\left(x_{k+1} + x_k\right)/2\right), N_4[u] = u'''\left(\left(x_{k+1} + x_k\right)/2\right).$$

i) Which of the three elements above are suitable for the following variational problem? Find $u \in H^1([a,b])$ such that

$$\int_a^b uv + u'v'\mathrm{d}x = \int_a^b fv \ \mathrm{d}x, \quad \forall v \in H^1([a,b]).$$

Justify your answer.

ii) Which of the three elements above are suitable for the following variational problem? Find $u \in H^2([a,b])$ such that

$$\int_a^b uv + u'v' + u''v''\mathrm{d}x = \int_a^b fv \ \mathrm{d}x, \quad \forall v \in H^2([a,b]).$$

Justify your answer.

Ex.7. Let V_h be a finite-dimensional subspace of a Hilbert space V, and let $u \in V$ and $u_h \in V_h$ satisfy

$$a(u,v) = \ell(v) \qquad \forall v \in V, \tag{V}$$
$$a(u_h, v_h) = \ell(v_h) \qquad \forall v_h \in V_h, \tag{V_h}$$

respectively, where $a(\cdot,\cdot)$ is a symmetric bilinear form on V and $\ell : V \to \mathbb{R}$ is a linear functional.

(a) Show that the Galerkin solution u_h is the best approximation to $u \in V$ when measured in the energy norm $\|\cdot\|_E$, that is,

$$\|u - u_h\|_E \leq \|u - v_h\|_E \qquad \forall v_h \in V_h,$$

where

$$\|v\|_E^2 = a(v,v), \qquad v \in V.$$

(b) Suppose that we have an enhanced approximation space W_h such that

$$V_h \subset W_h \subset V,$$

with associated solution $u_h^* \in W_h$ satisfying

$$a(u_h^*, v) = \ell(v) \qquad \forall v \in W_h.$$

Use the fact that

$$a(u - u_h^*, v) = 0 \qquad \forall v \in W_h$$

to show that

$$\|u - u_h\|_E^2 = \|u - u_h^*\|_E^2 + \|u_h^* - u_h\|_E^2.$$

Deduce that

$$\|u - u_h\|_E \geq \|u - u_h^*\|_E.$$

(c) Suppose that the enhanced space W_h can be written as

$$W_h = V_h \oplus Z$$

and that a strengthened Cauchy–Schwarz inequality holds:

$$|a(v_h, z_h)| \leq \gamma \, \|v_h\|_E \, \|z_h\|_E \qquad \forall v_h \in V_h, \ \forall z_h \in Z,$$

with $0 \leq \gamma < 1$.

Consider the simplified error representation problem: find $e_h \in Z$ satisfying

$$a(e_h, z_h) = \ell(z_h) - a(u_h, z_h) \qquad \forall z_h \in Z. \tag{Z_h^*}$$

Show that $e_h \in Z$ is equivalent to $u_h^* - u_h \in W_h$ and that

$$\|e_h\|_E^2 \leq \|u_h^* - u_h\|_E^2 \leq \frac{1}{1-\gamma^2} \, \|e_h\|_E^2, \qquad 0 \leq \gamma < 1.$$

Hint: To establish the left-hand inequality, use the fact that

$$a(u_h^* - u_h, z_h) = a(e_h, z_h) \qquad \forall z_h \in Z.$$

To establish the right-hand inequality, write

$$u_h^* - u_h = v_h + z_h, \qquad v_h \in X, \ z_h \in Z,$$

and use the inequality

$$a^2 + b^2 - 2\gamma ab \geq (1-\gamma^2)b^2.$$

2 Problems.

Pb. 1) (Raviart-Thomas element)

In this work we consider the Raviart-Thomas finite element RT_0:

- $\mathcal{T}_h$ is a triangular mesh of a smooth, bounded, closed domain $\Omega \subset \mathbb{R}^2$;
- K_i denotes the generic triangle of the mesh, for $i = 1, 2, \ldots, I$;
- e_j denotes its internal edges, for $j = 1, 2, \ldots, J$;
- d_l denotes its boundary edges, for $l = 1, 2, \ldots, L$.

Let $\mathrm{RT}_0(\mathcal{T}_h)$ be the space of the Raviart–Thomas finite element functions. The generic RT_0 function on a given triangle K can be written as

$$\varphi(x, y) = c \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \qquad a, b, c \in \mathbb{R},$$

and it satisfies the following properties:

(a) $(\nabla \cdot \varphi)|_K \in P_0(K)$,

(b) $\varphi \cdot n$ is constant on each edge of K.

We can thus define the global $\mathrm{RT}_0(\mathcal{T}_h)$ finite element space as:

$$\mathrm{RT}_0(\mathcal{T}_h) = \left\{ \varphi : \Omega \to \mathbb{R}^2 \;\middle|\; \varphi|_{K_i} \text{ is linear } \forall K_i \in \mathcal{T}_h, \text{ with continuous normal components over } e_j, d_l \right\} \tag{1}$$

RT_0 functions are therefore determined by the values of their normal components over the edges of the mesh, i.e. their degrees of freedom are

$$\varphi_i = (\varphi \cdot n)|_{e_i},$$

and the analogous quantities on the boundary edges.

Similarly, we can define the RT_0 interpolant of a function v as

$$I_{\mathrm{RT}_0}(v) = \varphi_h \in \mathrm{RT}_0(\mathcal{T}_h) : \quad (\varphi_h \cdot n)|_{e_i} = \frac{1}{|e_i|} \int_{e_i} v \cdot n, \qquad (\varphi_h \cdot n)|_{d_l} = \frac{1}{|d_l|} \int_{d_l} v \cdot n.$$

Finally, the global $\mathrm{RT}_0(\mathcal{T}_h)$ finite element functions belong to $H(\mathrm{div}, \Omega)$, as an immediate consequence of the following characterization.

2.1 Characterization of $H(\mathrm{div}, \Omega)$.

Let $\varphi : \Omega \to \mathbb{R}^2$ be such that:

(a) $\varphi|_{K_i} \in H^1(K_i)^2$ for each $K_i \in \mathcal{T}_h$;

(b) For each internal edge e_j let K^+, K^- be the two adjacent elements to e_j, with outward normals n^+, n^- respectively, $n^- = -n^+$. The traces of the normal components of φ on e_j, $\varphi|_{K^+} \cdot n^+$ and $\varphi|_{K^-} \cdot n^-$, are the same, i.e.

$$\varphi|_{K^+} \cdot n^+ + \varphi|_{K^-} \cdot n^- = 0.$$

Then $\varphi \in H(\mathrm{div}, \Omega)$.

Questions

Prove the following points:

(a) Prove property 1 of RT_0 elements, i.e.

$$(\nabla \cdot \varphi)|_K \in P_0(K).$$

(b) Prove property 2 of RT_0 elements, i.e. $\varphi \cdot n$ is constant on each edge of the triangle.
Hint: Let P_i, P_j be any two vertices of $K \in \mathcal{T}_h$. Write the expression of the line $r(t)$, $t \in \mathbb{R}$, passing through P_i, P_j, and verify that $\varphi|_{r(t)} \cdot n$ is a constant quantity.

(c) Compute the expression of the RT_0 basis functions on the reference triangle.

2.2 Darcy's Problem

Consider the Darcy equations for flows in porous media in the square domain

$$\Omega = [0,1]^2$$

$$\begin{cases} -k\nabla p = u, & \text{in } \Omega, \\ \nabla \cdot u = f, & \text{in } \Omega, \\ p = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $k : \Omega \to \mathbb{R}$ is such that

$$0 < k_{\min} \leq k(x,y) \leq k_{\max} < \infty, \qquad \forall (x,y) \in \Omega.$$

We can combine the first and the second equations to obtain an equivalent elliptic problem:

$$\begin{cases} -\nabla \cdot (k\nabla p) = f, & \text{in } \Omega, \\ p = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

(a) Let $k = 1$ and

$$p(x,y) = \sin(\pi x)\sin(\pi y).$$

Derive the expression of u and compute f so that the above equations are satisfied.

(b) Solve the mixed problem (2) using an RT_0 finite element discretization for u and a P_0 discretization for p. Recall the expected convergence rate and verify your results.

FreeFem hint:

```
fespace Vh(Th,RT0);
fespace Qh(Th,P0);
Vh [uh1, uh2],[tauh1, tauh2];
Qh ph,qh;
problem Darcy([uh1,uh2,ph],[tauh1,tauh2,qh]) = ...
```

(c) Solve the elliptic problem (3) using a P_1 discretization for p. Recall the convergence rate for the L^2 norm of the error and verify your results.

(d) Compare the two error convergence curves for p in the same plot and comment the results you obtain. Which approximation is the most accurate?

(e) Reconstruct a numerical approximation for $u \in P_0$ starting from the P_1 approximation of p in (3). What rate of convergence do you expect for the L^2 error on u?

Hint: In FreeFem it is possible to directly obtain a numerical reconstruction of $u \in P_0$ from the computed P_1 approximation of p by using the commands `dx()` and `dy()`.

(f) Compare the two error convergence curves for u (i.e. the one obtained with the mixed formulation and the one reconstructed from the elliptic approximation) on the same plot and comment the results you obtain. Which approximation is the most accurate?

Pb. 2) (Creuzeix-Raviart element) Consider the unit square

$$\Omega = (0,1)^2.$$

Let $\mathcal{T}_h$ be a regular mesh for Ω, F_h be the set of edges of $\mathcal{T}_h$, and F_h^{int} be the set of interior edges of $\mathcal{T}_h$. For each edge $F \in F_h^{\text{int}}$, let us denote with $K_{F,+}$ and $K_{F,-}$ the two elements $K \in \mathcal{T}_h$ separated by F.

Moreover, let us consider a function $v : \Omega \to \mathbb{R}$ and its restrictions on $K_{F,+}$ and $K_{F,-}$, that we denote as $v_{F,+}(x)$ and $v_{F,-}(x)$ respectively. If v is a continuous function over Ω, its value on each edge F is uniquely defined; if instead we only require continuity separately in each triangle $K \in \mathcal{T}_h$, the value of $v_{F,+}$ and $v_{F,-}$ on F may differ. Such difference is called the *jump* of v across F and it is denoted by $[v]_F$, i.e.

$$[v]_F(x) = v_{F,+}(x) - v_{F,-}(x), \qquad x \in F.$$

The Crouzeix–Raviart finite element space is the space of piecewise linear functions whose jumps over internal edges have zero mean:

$$V_h = \left\{ v_h \in L^2(\Omega) : v_h|_K \in \mathbb{P}_1(K)\ \forall K \in \mathcal{T}_h, \text{ and } \int_F [v_h] = 0\ \forall F \in F_h^{\text{int}} \right\}.$$

(a) Show that Crouzeix–Raviart functions are continuous in the midpoint of each internal edge (see Figure **??**).

Hint: Show the continuity in the midpoint by applying the midpoint quadrature rule to the condition $\int_F [v_h] = 0$.

(b) Let a_F denote the midpoint of the edge $F \in F_h$. Show that the values $v_h(a_j)$ uniquely identify a function in V_h.

(c) Draw the basis functions associated to the generic degree of freedom.

(d) Let $I_{h,\text{ref}}^{CR} : H^2(\widehat{K}) \to V_h(\widehat{K})$ be the interpolant operator on the reference triangle:

$$I_{h,\text{ref}}^{CR}(\widehat{u})(x) = \sum_{j=1}^{3} \widehat{u}(a_j)\, \widehat{\varphi}_j(x).$$

Observe that from here on, quantities related to the reference triangle will be denoted by a hat. Show that $I_{h,\text{ref}}^{CR}$ is continuous from $H^2(\widehat{K})$ to $V_h(\widehat{K})$ with $H^1(\widehat{K})$ norm.

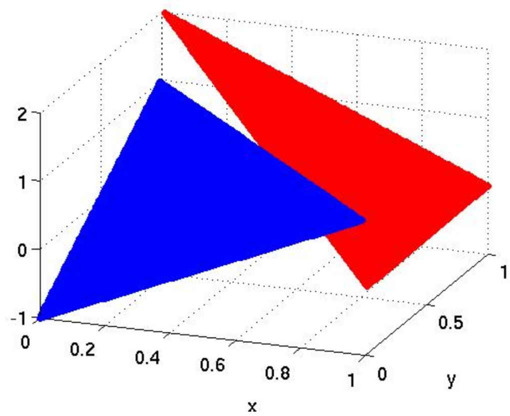

Figure 1: Crouzeix–Raviart functions are linear in each triangle and discontinuous among triangles but continuous at the midpoints of internal edges.

(a) Let $I_h^{CR} : H^2(\Omega) \to V_h$ be the global interpolant operator

$$I_h^{CR}(u)(x) = \sum_{F \in F_h} u(a_F)\, \varphi_F(x).$$

Prove that

$$\sum_{K \in \mathcal{T}_h} |u - I_h^{CR}(u)|_{H^1(K)} \le Ch|u|_{H^2(\Omega)},$$

$$\|u - I_h^{CR}(u)\|_{L^2(\Omega)} \le Ch^2 |u|_{H^2(\Omega)}.$$

Hint: Work on each element. You will need the following properties:

i. The seminorm transformation lemma: for all integer $m \ge 0$ and all $v \in H^m(K)$, $\widehat{v} \in H^m(\widehat{K})$, there exists a constant $C = C(m) > 0$ such that

$$|\widehat{v}|_{H^m(\widehat{K})} \le C \|B_K\|^m |\det B_K|^{-1/2} |v|_{H^m(K)},$$

$$|v|_{H^m(K)} \le C \|B_K^{-1}\|^m |\det B_K|^{1/2} |\widehat{v}|_{H^m(\widehat{K})},$$

where B_K is the matrix of the linear mapping from $\widehat{K}$ to K.
Moreover,

$$\|B_K\| \le \frac{h_K}{\widehat{\rho}}, \qquad \|B_K^{-1}\| \le \frac{\widehat{h}}{\rho_K}.$$

ii. The fact that $I_h^{CR}(u) = I_{h,\mathrm{ref}}^{CR}(\widehat{u})$.

iii. $v_h = I_h^{CR}(v_h)$ for $v_h \in V_h$.

iv. The continuity of $I_{h,\mathrm{ref}}^{CR}$ on the reference element.

v. The Deny–Lions lemma: for every $r \geq 0$ let $\mathbb{P}_r(\widehat{K})$ be the set of polynomials of degree r over $\widehat{K}$. Then there exists a constant $C = C(r, \widehat{K})$ such that

$$\inf_{\widehat{p} \in \mathbb{P}_r(\widehat{K})} \|\widehat{p} + \widehat{v}\|_{H^{r+1}(\widehat{K})} \leq C|\widehat{v}|_{H^{r+1}(\widehat{K})}, \qquad \forall \widehat{v} \in H^{r+1}(\widehat{K}).$$

vi. For a regular mesh the ratio h_K/ρ_K is uniformly bounded by a constant independent of h:

$$\frac{h_K}{\rho_K} \leq \delta < \infty.$$

3 Condition Number of the Galerkin Matrix

Consider the Laplacian problem on the unit square:

$$\begin{cases} -\Delta u = f & \text{in } \Omega = (0,1)^2, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

(a) Using FreeFem++, compute the Galerkin matrix for this problem and save it to file. Use the following commands:

- `varf a(u,v)= int2d(..)(..)  + on(..);`
- `matrix A = a(Vh,Vh);`
- `ofstream matrixfile("A.dat");`
- `matrixfile << A << endl;`

(b) Compute the condition number using `cond(A)`.

(c) Repeat for $n = 5, 10, 20, 40$ elements per side. Fill the table:

n	$h = 1/n$	$\text{cond}(A_h)$
5	–	–
10	–	–
20	–	–
40	–	–

(d) Repeat the analysis for a non-structured mesh built with `border` and `buildmesh`. Compare with the structured case.

(e) Consider a grid refined toward the boundaries:

```
mesh Th = square(n,n);
Th = movemesh(Th,[(1-cos(pi*x))/2.,(1-cos(pi*y))/2.]);
```

Repeat the analysis. What happens?