# Homework 3: A Posteriori Error Estimation and Flux Approximation

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## 1 Problem Statement

Let $\Omega$ be a bounded regular open set in $\mathbb{R}^d (d = 1, 2, 3)$, with $f \in L^2(\Omega)$, and constants $k, \alpha > 0$. We consider the partial differential equation

$$\begin{cases} -\operatorname{div}(k\nabla u) + \alpha u = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The associated variational formulation is to find $u \in V := H_0^1(\Omega)$ such that for all $v \in V$,

$$\int_\Omega k\nabla u \cdot \nabla v + \alpha uv \, dx = \int_\Omega fv \, dx.$$

To determine a numerical approximation of $u$, we use a Galerkin-type method. In other words, we consider a finite-dimensional subspace $V_h$ of $V$ and denote by $u_h$ the element of $V_h$ such that for all $v_h \in V_h$,

$$\int_\Omega k\nabla u_h \cdot \nabla v_h + \alpha u_h v_h \, dx = \int_\Omega fv_h \, dx.$$

We then introduce the approximation error $e_h = u_h - u$ and use certain a priori estimates. Specifically, if we use the Lagrange P1 finite element method, there exists a constant $C$ independent of $f$ such that if the potential $u$ is regular and $h$ is the mesh size (assumed regular) of $\Omega$, we have

$$\|e_h\|_V \leq Ch\|u\|_{H^2(\Omega)},$$

where $\|\cdot\|_V$ denotes the energy norm, defined as

$$\|v\|_V = \left( \int_\Omega k\nabla v \cdot \nabla v + \alpha|v|^2 \, dx \right)^{1/2}.$$

Unfortunately, we do not know the solution $u$ a priori. Hence, the previous estimate only provides an order of magnitude of the error. We aim to bound the error by a quantity that only involves known or explicitly computable data.

### Question 1: A Posteriori Estimation

The approximation error $e_h$ itself satisfies a variational problem. Indeed, $e_h \in V$ is such that for all $v \in V$,

$$\int_\Omega k\nabla e_h \cdot \nabla v + \alpha e_h v \, dx = \int_\Omega k\nabla u_h \cdot \nabla v + \alpha u_h v \, dx - \int_\Omega fv \, dx.$$

Show that for all $\sigma \in H(\mathrm{div}) := \left\{ \tau \in L^2(\Omega)^n \mid \nabla \cdot \tau \in L^2(\Omega) \right\}$, we have

$$\frac{1}{2}\|e_h\|_V^2 \leq -G_h(\sigma),$$

where

$$G_h(\sigma) = -\frac{1}{2} \int_\Omega k^{-1}|\sigma - k\nabla u_h|^2 \, dx - \frac{1}{2} \int_\Omega \alpha^{-1}|f - \alpha u_h + \nabla \cdot \sigma|^2 \, dx.$$

### Question 2: Flux Approximation

We aim to optimize the second term of inequality (2) over a space $W_h$. Specifically, we introduce $\sigma_h \in W_h \subset H(\text{div})$ such that

$$G_h(\sigma_h) = \max_{\tau \in W_h} G_h(\tau).$$

Show that $\sigma_h \in W_h$ is such that for all $\tau \in W_h$,

$$\int_\Omega k^{-1}\sigma_h \cdot \tau \, dx + \int_\Omega \alpha^{-1}(f + \nabla \cdot \sigma_h)\nabla \cdot \tau \, dx = 0.$$

Deduce that $\sigma_h$ is an approximation of the flux $\sigma = k\nabla u$.

## 2 One-Dimensional Case

We consider the case $\Omega = (0,1)$ and use a finite element discretization to determine an approximation $u_h$ of the potential $u$ and an approximation $\sigma_h$ of the flux $\sigma$. Specifically, we partition $\Omega$ into $N+1$ intervals $(x_i, x_{i+1})$ for $i = 0, \ldots, N$, where $x_i = ih$ and $h = 1/(N+1)$. We introduce the finite element space $V_h$ of P1 elements on $(0,1)$ that vanish on the boundary.

### Question 3: Optimality Conditions for the Potential

Show that $U_h$ is the solution of the linear system

$$A_h U_h = b_h,$$

where $A_h$ is the matrix $A_h = kh^{-1}K + \alpha h M$, with $K$, $M$ being two matrices and $b$ a vector to be determined.

### Question 4: Potential Calculation

Numerically calculate the solution of (3) for a constant function $f$. Plot the graph of $u$. Numerical application: $N = 100$, $f = 1$, $k = 1$, and $\alpha = 1$.

Following the same approach, we compute the approximation of the flux $\Sigma_h \in \mathbb{R}^{N+1}$. We choose the approximation space for the flux $\sigma_h$ as the set of P1 finite elements,

$$W_h = \{\sigma_h \in H^1(\Omega) \mid v_h|_{(x_i,x_{i+1})} \in P1 \text{ for all } i = 0, \ldots, N\}.$$

Let $\Sigma_h$ denote the coordinates of $\sigma_h$, the Galerkin approximation of $\sigma_h$ on $W_h$.

### Question 5: Optimality Conditions for the Flux

Show that $\Sigma_h$ is the solution of the system

$$B_h \Sigma_h = (k^{-1}hM' + \alpha^{-1}h^{-1}K')\Sigma_h = c_h,$$

with $K'$, $M'$ being two matrices and $c_h$ a vector to be determined.

### Question 6: Flux Calculation

Numerically compute the solution of (4). Plot the graph of $\sigma_h$ using the same numerical values as in the previous question.

## Question 7: Discretization of the Injection Operator $V_h$ into $W_h$

7.a) Show that the matrix $I_V$, which associates the coordinates $U_h$ of $u_h$ in the basis of $V_h$ to its coordinates in $W_h$, is the $(N+2) \times N$ matrix of the form:

$$I_V = \begin{pmatrix} 0 \\ \mathrm{Id} \\ 0 \end{pmatrix}$$

7.b) Write a freefem++ code that implements the matrix $I_V$.

## Question 8: Discretization of the Injection Operator $W_h$ into $X_h$

8.a) Show that the matrix $I_W$, which associates the coordinates $\Sigma_h$ of $\sigma_h$ in $W_h$ to its coordinates in $X_h$, is the $2(N+1) \times (N+2)$ matrix of the form:

$$I_W = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & c & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & c & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad \text{with } c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

8.b) Write a freefem++ code that implements the matrix $I_W$.

## Question 9: Discretization of the Gradient Operator on $X_h$

Let $v_h$ be an element of $X_h$ and $\tau_h \in X_h$ defined for all $i = 0, \cdots, N$ by:

$$\tau_h|_{(x_i,x_{i+1})} = \nabla v_h|_{(x_i,x_{i+1})}.$$

Denoting by $V$ and $T$ the respective coordinates of $v_h$ and $\tau_h$ in $X_h$,

9.a) Show that:

$$T = D_h V,$$

where $D_h$ is the $2(N+1) \times 2(N+1)$ matrix:

$$D_h = h^{-1} \begin{pmatrix} D & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & D \end{pmatrix}, \quad \text{with } D = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

9.b) Write a freefem++ code that implements the matrix $D_h$.

## Question 10: Discretization of the Mass Operator on $X_h$

Let $\tau_h$ be an element of $X_h$ with coordinates $T$.

10.a) Show that:

$$\int_\Omega |\tau_h|^2 \, dx = N_h T \cdot T,$$

where:

$$N_h = \frac{h}{6} \begin{pmatrix} N & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & N \end{pmatrix}, \quad \text{with } N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

10.b) Write a freefem++ code using varf that implements the matrix $N_h$.

### Question 11: Error Estimation

To slightly simplify the analysis, assume $f \in X_h$. Let $F_h$ be the coordinates of $f$ in $X_h$. Deduce from the estimation (2) that:

$$\begin{aligned}2\|e_h\|_V \leq\, & k^{-1} N_h \left(I_W \Sigma_h - k D_h I_W I_V U_h\right) \cdot \left(I_W \Sigma_h - k D_h I_W I_V U_h\right) \\ & + \alpha^{-1} N_h \left(F_h - \alpha I_W I_V U_h + D_h I_W \Sigma_h\right) \cdot \left(F_h - \alpha I_W I_V U_h + D_h I_W \Sigma_h\right).\end{aligned}$$

Using this estimation, calculate an upper bound for the error $\|e_h\|_V$ in the calculation of $u$. Use the same data as in the previous questions, with $N = 100$ and $N = 1000$.

## 3 Two-Dimensional Case

Consider now the two-dimensional case with the unit disk as the domain $\Omega$.

### Question 12: Potential Calculation

Using FreeFem++, find an approximation of the potential using the P1 finite element method. Use a mesh with a density $\delta_n$ of elements per unit length with $\delta_n = 10$. Also, choose $f = 1$, $\alpha = 1$, and $k = 1$. Plot the isovalues of the potential and provide the average of $u_h$ on $\Omega$.

### Question 13: Flux Calculation

Using FreeFem++, find an approximation of the flux with Raviart-Thomas elements of degree zero (RT0 in FreeFem++). Use the same mesh and data as in the previous question. Plot the resulting field $\sigma_h$ on a graph.

### Question 14: Error Estimation

Determine an upper bound on the error $\|e_h\|_V$. Use the same mesh and data as in the previous two questions.

### Question 15: Convergence Rate

Plot the graph, in logarithmic coordinates, of the error $\|e_h\|_V$ as a function of the density $\delta_n$ of points on the boundary per unit length (choose $\delta_n$ from 10 to 100). What is the observed convergence rate? Compare the obtained rate to the theoretical convergence rate of the error.