Master Maths
First year

2025-2026
Kasdi Merbah University

Semismooth Newton Method
Homework 3

Ex.1 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = |x|^p \quad \text{for } p > 2.$$

(a) Consider the Newton method for finding first-order stationary points of f. Initialize with $x_0 > 0$.

(b) Show that the method converges to the unique minimum of f.

Ex.2 Let $f(x) = |x|..$

(a) Show that this function is Lipschitz continuous but not differentiable at $x = 0$.

(b) Compute the Clarck generalized Jacobian of f.

(c) Show that that f is semismooth.

Ex.3 Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \max(0, x) = \begin{cases} 0, & x < 0, \\ x, & x \geq 0. \end{cases}$$

(a) Show that this function is Lipschitz continuous but not differentiable at $x = 0$.

(b) Compute the Clark generalized Jacobian of f.

(c) Verify that f is semismooth.

Ex.4 Consider $f(x) = \sqrt{|x|}$. Show that f is not semismooth.

Ex.5 Let

$$f(x) = \begin{cases} x^2, & x < 0, \\ x, & x \geq 0. \end{cases}$$

Show that f is semismooth.

Ex.6 Define for bounded $\Omega \subset \mathbb{R}^d$ and bounded convex subset $U_{ad} \subset L^2(\Omega)$ the projection operator

$$\text{proj}_{U_{ad}} : L^2(\Omega) \to L^2(\Omega), \qquad \text{proj}_{U_{ad}}(v) = \arg\min_{u \in U_{ad}} \frac{1}{2}\|u - v\|^2_{L^2(\Omega)}.$$

(a) Show that the minimization problem has a unique solution.

(b) Assume now that

$$U_{ad} = \{u \in L^2(\Omega) : \alpha(x) \le u(x) \le \beta(x) \text{ for a.e. } x \in \Omega\},$$

for functions $\alpha, \beta \in L^\infty(\Omega)$. Show that

$$\mathrm{proj}_{U_{ad}}(u)(x) = \begin{cases} \beta(x) & \text{if } u(x) \ge \beta(x), \\ u(x) & \text{if } u(x) \in (\alpha(x), \beta(x)), \\ \alpha(x) & \text{if } u(x) \le \alpha(x), \end{cases}$$

for almost every $x \in \Omega$.

Ex.7 Let $\Omega \subset \mathbb{R}^n$ be bounded and set $X = L^\infty(\Omega)$. Consider the operator

$$G : u \mapsto \mathcal{L}(X), \qquad G(u)[h](x) = \begin{cases} h(x) & \text{if } u(x) \ge 0, \\ 0 & \text{if } u(x) < 0. \end{cases}$$

Show that $G : u \mapsto \mathcal{L}(X)$ is not the Newton derivative of the function

$$F : X \to X, \qquad F(u)(x) = \max(0, u(x)).$$

Ex.8 Let $\Omega \subset \mathbb{R}^2$ be a bounded region with piecewise smooth boundary. Let $X := L^1(\Omega, \mathbb{R})$. We consider the following problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f : X \to \mathbb{R}$ is a continuous function. Show that problem (1) is equivalent to a nonsmooth integral equation of the form:

$$u - H(u) = 0$$

Ex.9 Consider the problem:

$$\begin{cases} -\Delta u = \dfrac{\pi^2}{2} \max(0, u - 1) & \text{in } \Omega = (0,1) \times (0,1), \\ u = \phi(x, y) & \text{on } \partial\Omega, \end{cases}$$

where

$$\phi(0, \xi) = \phi(\xi, 0) = 1 + 2\cos\left(\frac{\pi}{2}\xi\right), \qquad \phi(1, \xi) = \phi(\xi, 1) = 1 - \pi\xi.$$

This problem has an exact solution

$$u(x, y) = \begin{cases} 1 + 2\cos\left(\dfrac{\pi}{2}(x + y)\right), & x + y \le 1, \\ 1 + \pi(1 - x - y), & \text{otherwise.} \end{cases}$$

Write a Freefem++ code to solve this problem using Newton-Kantorivich algorithm.