Master Maths
Second year

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Work Lab: Flux reconstruction.

Preliminaires

Let $\Omega \subset \mathbb{R}^2$ be a polygon with Lipschitz boundary $\partial\Omega = \Gamma_{\mathrm{D}} \cup \Gamma_{\mathrm{N}}$. We consider the following model problem : for a given source term $f \in L^2(\Omega)$ and a given prescribed data g on the Dirichlet part of the boundary Γ_{D}, find $u : \Omega \to \mathbb{R}$ such that :

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \quad u = g \text{ on } \Gamma_{\mathrm{D}} \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{N}}. \end{cases} \tag{1}$$

The weak solution of problem (1) is a function $u \in H^1(\Omega)$ such that $u|_{\Gamma_{\mathrm{D}}} = g$ and

$$(\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega) \tag{2}$$

Let $\mathcal{T}_h$ be a triangulation of Ω. The finite element method seeks for an approximate solution u_h to the exact solution u in a finite-dimensional subspace V_h^p of $H^1(\Omega)$. For a polynomial degree $p \geq 1$, we namely consider

$$V_h^p := \left\{ v_h \in H^1(\Omega), \, v_h|_K \in \mathcal{P}_p(K) \quad \forall K \in \mathcal{T}_h \right\} = \mathcal{P}_p\left(\mathcal{T}_h\right) \cap H^1(\Omega) \tag{3}$$

Above, $\mathcal{P}_q(K)$ stands for the space of polynomials of total degree at most $q \geq 0$ on the mesh element $K \in \mathcal{T}_h$ and $\mathcal{P}_q\left(\mathcal{T}_h\right)$ denotes piecewise q-degree polynomials with respect to the mesh $\mathcal{T}_h$. Note that by the inclusion in $H^1(\Omega)$, the functions in V_h^p have their traces continuous over all mesh faces. Then $u_h \in V_h^p$ needs to satisfy $u_h|_{\Gamma_{\mathrm{D}}} = g$ and

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall v_h \in V_h^p \text{ such that } \; v_h|_{\Gamma_{\mathrm{D}}} = 0 \tag{4}$$

We take Ω is a unit square, $\Gamma_D = \partial\Omega, \Gamma_{\mathrm{N}} = \emptyset$, $g = 0$, and

$$f = -2\left(x^2 + y^2\right) + 2(x + y).$$

the exact solution is

$$u(x, y) = x(x-1)y(y-1) \tag{5}$$

which is smooth, $u \in C^\infty(\bar{\Omega})$.

Exercise 1 (Finite element code)

1. **1.** Specify the user input in the Freefem++ script :
   int nds = 10 ; // number of mesh points on one domain unity edge
2. **2.** Specify the exact solution u together with its derivatives, the right-hand side f, and the Dirichlet boundary datum g. This is done in the section exact solution and its derivatives.
3. **3.** Generate a triangular mesh $\mathcal{T}_h$ of Ω. In Freefem++, this is achieved via the command mesh Th = square(nds,nds) ;
4. **4.** Define the space V_h^p from (3) in Freefem++ : this is done via the command : fespace Vh(Th,Pcont) ;
5. **5.** Compute the finite element approximation u_h by (4).
6. **6.** Plot the exact solution u and its finite element approximation u_h.
7. **7.** Plot the flux of the exact solution given by $-\nabla u$ and the flux of the finite element approximation given by $-\nabla u_h$.[1]
   a. Choose some two neighboring mesh elements and plot the details.
   b. What do you observe ?
   c. Does the exact flux $-\nabla u$ seem to be continuous across the mesh faces, or at least to have the normal component $-\nabla u \cdot \boldsymbol{n}_F$ continuous across any mesh face F ? (Here, $\boldsymbol{n}_F$ is a unit normal vector of F.)[2].
   d. What about the flux approximation $-\nabla u_h$? Please inspect various polynomial degrees $1 \leq p \leq 4$. $-\nabla u_h$ for $p = 1$ is a piecewise constant, discontinuous, vector-valued field. This in particular means that, for a given mesh face F, it is not true that "flows out" from one mesh element sharing F across F "flows in" the other mesh element sharing F ; the approximate flux $-\nabla u_h$ is unphysical, non-conservative. The exception is only the case $p = 4$: since the exact solution is here a polynomial of order 4, we actually in this case have $u_h(x, y) = u(x, y) = x(x - 1)y(y - 1)$.

1. In the FreeFem++ graphics window, the size of the arrows is modified by pressing "a" and "A".
2. Please notice that the latter, weaker, property, means that, for a given mesh face F, what "flows out" from one mesh element sharing F across F "flows in" the other mesh element sharing F

Exercise 2 (Flux reconstruction by averaging)

Let

$$\boldsymbol{V}_h^{p'} := \{\boldsymbol{v}_h \in \boldsymbol{H}(\mathrm{div}, \Omega), \boldsymbol{v}_h|_K \in \boldsymbol{\mathcal{RT}}_{p'}(K) \quad \forall K \in \mathcal{T}_h\} = \boldsymbol{\mathcal{RT}}_{p'}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}, \Omega) \tag{6}$$

be the Raviart-Thomas space of degree $p' \geq 0$. Here,

$$\boldsymbol{\mathcal{RT}}_{p'}(K) = [\mathcal{P}_{p'}(K)]^d + \boldsymbol{x}\mathbb{P}_{p'}(K)$$

is the Raviart-Thomas space on a single mesh element $K \in \mathcal{T}_h$ and $\mathcal{RT}_{p'}(\mathcal{T}_h)$ is the space of all functions that belong to $\boldsymbol{\mathcal{T}}_{p'}(K)$ on each mesh element, the so-called broken Raviart-Thomas space. The inclusion into $\boldsymbol{H}(\mathrm{div}, \Omega)$ ensures that all functions from the space $\boldsymbol{V}_h^{p'}$ have their normal trace continuous over all mesh faces. We usually set the degree p' to p or to $p-1$, i.e., equal to that of the finite element approximation u_h ore one less.

1. Implement a flux reconstruction $\boldsymbol{\sigma}_h$ in the Raviart-Thomas space $\boldsymbol{V}_h^{p'}$ by averaging. This idea is to start from $-\nabla u_h$ and to use a simple averaging of the values that $-\nabla u_h$ takes in the degrees of freedom of Raviart-Thomas space $\boldsymbol{V}_h^{p'}$ i.e,

$$\boldsymbol{\sigma}_h(Dof) = \text{mean value of all } (-\nabla u_h)(Dof) \tag{7}$$

2. Plot the reconstructed flux $\boldsymbol{\sigma}_h$. What do you observe ?
3. Plot the misfit of the optimal divergence of the reconstructed flux $\boldsymbol{\sigma}_h$. More precisely, the goal is to compute the following L^2 norms on each mesh element $K \in \mathcal{T}_h$:

$$\|\Pi_{p'} f - \nabla \cdot \boldsymbol{\sigma}_h\|_K \tag{8}$$

where $\Pi_{p'}$ is the $L^2(\Omega)$-orthogonal projection onto discontinous piecewise polynomials of degree p' of the space $\mathcal{P}_{p'}(\mathcal{T}_h)$, i.e., $\Pi_{p'} f \in \mathbb{P}_{p'}(\mathcal{T}_h)$ is such that $(\Pi_{p'} f, v_h) = (f, v_h)$ for all $v_h \in \mathbb{P}_{p'}(\mathcal{T}_h)$, or, still equivalently, $(\Pi_{p'} f, v_h)_K = (f, v_h)_K$ for all $v_h \in \mathbb{P}_{p'}(K)$ and for all mesh elements $K \in \mathcal{T}_h$. For an equilibrated flux, the quantities in (8) would be zero. What do you observe here ?

Exercise 3 (Flux reconstruction by equilibration)

Let the Raviart-Thomas space of degree $p' \geq 0$ be given by (6).

1. Implement the equilibrated flux reconstruction $\boldsymbol{\sigma}_h$ in the Raviart-Thomas space $\boldsymbol{V}_h^{p'}$. Let $-\nabla u_h$ be computed. For each fixed mesh vertex $\boldsymbol{a} \in \mathcal{V}_h$, let $\mathcal{T}_{\boldsymbol{a}}$ be the patch of all mesh elements from $\mathcal{T}_h$ that share the vertex $\boldsymbol{a}$ and $\omega_{\boldsymbol{a}}$ the corresponding patch subdomain. Let ψ^a be the hat function, i.e., the unique continuous and piecewise 1 -st order polynomial that takes the value 1 in the vertex $\boldsymbol{a}$ and the value 0 in all other mesh vertices; note that the support of ψ^a is the patch subdomain ω_a. For a vertex $\boldsymbol{a}$ inside the computational domain Ω, let $\boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ be the subspace of all functions from $\boldsymbol{H}(\mathrm{div}, \omega_{\boldsymbol{a}})$ whose normal trace vanishes on $\partial\omega_{\boldsymbol{a}}$. For a vertex $\boldsymbol{a}$ on the boundary of Ω, we only request the normal trace to vanish on 1) the part of $\partial\omega_a$ where $\psi^{\boldsymbol{a}}$ is zero (typically the part of $\partial\omega_{\boldsymbol{a}}$ not contained in $\partial\Omega$); and 2) $\Gamma_{\mathrm{N}} \cap \partial\omega_{\boldsymbol{a}}$. The local equilibration has two stages : first we need to solve the local quadratic minimization problem

$$\boldsymbol{\sigma}_h^a := \arg \min_{\substack{\boldsymbol{v}_h \in \mathcal{RR}_{p'}(\mathcal{T_T}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_a) \\ \nabla \cdot \boldsymbol{v}_h = \Pi_{p'}(f\psi^a - \nabla u_h \cdot \nabla \psi^a)}} \left\| \psi^a \nabla u_h + \boldsymbol{v}_h \right\|_{\omega_a}^2 \tag{10a}$$

for all $\boldsymbol{a} \in \mathcal{V}_h$. Then we run over all mesh vertices $\boldsymbol{a} \in \mathcal{V}_h$ and sum the individual contributions $\boldsymbol{\sigma}_h^a$ as

$$\boldsymbol{\sigma}_h := \sum_{a \in \mathcal{V}_h} \boldsymbol{\sigma}_h^a \tag{10b}$$

Evoking the Euler-Lagrange optimality conditions of (10a), (10a) can be equivalently written as : find $\boldsymbol{\sigma}_h^a \in \boldsymbol{\mathcal{R}}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ with $\nabla \cdot \boldsymbol{\sigma}_h^{\boldsymbol{a}} = \Pi_{p'}(f\psi^{\boldsymbol{a}} - \nabla u_h \cdot \nabla \psi^a)$ such that

$$(\boldsymbol{\sigma}_h^a, \boldsymbol{v}_h)_{\omega_a} = -(\psi^a \nabla u_h, \boldsymbol{v}_h)_{\omega_a} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{R}T}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_a) \text{ with } \nabla \cdot \boldsymbol{v}_h = 0 \tag{11}$$

One could now implement (11), but one would need for this purpose to construct a basis of the Raviart-Thomas space of piecewise polynomial vector-valued fields from $\boldsymbol{\mathcal{R}}\mathcal{T}_{p'}(\mathcal{T}_{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$ with the property $\nabla \cdot \boldsymbol{v}_h = 0$. To avoid this, we further rewrite equivalently (11) as : find $\boldsymbol{\sigma}_h^a \in \boldsymbol{\mathcal{R}}\mathcal{T}_{p'}(\mathcal{T}_a) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_a)$, together with the additional scalar-valued piecewise polynomial $\gamma_h^a \in \mathbb{P}_{p'}(\mathcal{T}_a)$, such that

$$(\boldsymbol{\sigma}_h^a, \boldsymbol{v}_h)_{\omega_a} - (\gamma_h^a, \nabla \cdot \boldsymbol{v}_h)_{\omega_a} = -(\psi^a \nabla u_h, \boldsymbol{v}_h)_{\omega_a} \quad \forall \boldsymbol{v}_h \in \boldsymbol{\mathcal{R}}\mathcal{T}_{p'}(\mathcal{T}_a) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_a), \tag{12a}$$

$$(\nabla \cdot \boldsymbol{\sigma}_h^a, q_h)_{\omega_a} = (f\psi^a - \nabla u_h \cdot \nabla \psi^a, q_h)_{\omega_a} \quad \forall q_h \in \mathbb{P}_{p'}(\mathcal{T}_a). \tag{12b}$$

2. Plot the finite element flux $-\nabla u_h$, the hat-function-weighted finite element flux $-\psi^a \nabla u_h$, the equilibrated flux contribution $\boldsymbol{\sigma}_h^a$, and the hat-function-weighted exact flux $-\psi^a \nabla u$ on each patch subdomain ω_a. Describe what you observe : differences and similarities between the plots, sizes of these vector fields close to the vertex $\boldsymbol{a}$ and close to the boundary of the patch subdomain $\omega_{\boldsymbol{a}}$ (not shared by the boundary $\partial\Omega$), continuity across the mesh faces, and normal component continuity across the mesh faces. (Attention, FreeFem++ mainly distinguishes the sizes of vector fields by color and not by size.)

3. Plot the reconstructed flux $\boldsymbol{\sigma}_h$. What do you observe ?

4. Plot the divergence misfit of the reconstructed flux $\boldsymbol{\sigma}_h$. More precisely, the idea is to compute the elementwise L^2 norms (8). From definition (10), these should be zero. What do you observe here ?

Exercise 4 (Error)

We will compute here the errors between the exact solution u of (2) and its finite element approximation u_h of (4).

1. Compute the error $\|\nabla(u - u_h)\|$, as well as its elementwise contributions

$$\|\nabla(u - u_h)\|_K \tag{13}$$

for each mesh element $K \in \mathcal{T}_h$.
2. Plot the elementwise error contributions (13).

Exercise 5 (A posteriori error estimators by equilibrated fluxes)

We will now compute the a posteriori error estimators on the error between the exact solution u of (2) and its finite element approximation u_h of (4). We start by the equilibrated fluxes of Exercice 3, in the setting with $p' = p$ according to the theory developed in the lectures. Recall that in this case, we have

$$\|\nabla(u - u_h)\| \leq \eta_h := \left\{ \sum_{K \in \mathcal{T}_h} \left[\|\nabla u_h + \boldsymbol{\sigma}_h\|_K + \frac{h_K}{\pi} \|f - \Pi_{p'} f\|_K \right]^2 \right\}^{\frac{1}{2}} \tag{14}$$

1. Plot the elementwise a posteriori error estimators $\left[\|\nabla u_h + \boldsymbol{\sigma}_h\|_K + \frac{h_K}{\pi} \|f - \Pi_{p'} f\|_K\right]$. Compare them to the plots of the elementwise errors from Exercice 4. What do you observe ?
2. Plot the "data oscillation" part of the estimators given by $\frac{h_K}{\pi} \|f - \Pi_{p'} f\|_K$.
3. Compare the size of the a posteriori error estimator η_h to the size of the error $\|\nabla(u - u_h)\|$. This is best done in terms of the so-called effectivity index

$$I_{\text{eff},h} := \frac{\eta_h}{\|\nabla(u - u_h)\|} \tag{15}$$

What do you observe ?

Exercise 6 (A posteriori error estimators by averaged fluxes)

We will now also compute the a posteriori error estimators for the averaged fluxes of Exercice 2. In this case, there is no guaranteed upper bound, though we may still hope to obtain

$$\|\nabla(u - u_h)\| \lesssim \|\nabla u_h + \boldsymbol{\sigma}_h\| \tag{16}$$

1. Plot the elementwise a posteriori error estimators $\|\nabla u_h + \boldsymbol{\sigma}_h\|_K$. Compare them to the plots of the elementwise errors from Exercice 4. What do you observe ?
2. Compare the size of the a posteriori error estimator $\|\nabla u_h + \boldsymbol{\sigma}_h\|$ to the size of the error $\|\nabla(u - u_h)\|$. This is best done in terms of the so-called effectivity index

$$I_{\text{eff},h} := \frac{\|\nabla u_h + \boldsymbol{\sigma}_h\|}{\|\nabla(u - u_h)\|} \tag{17}$$

What do you observe ?