MEF pour les inéquations variationnelles

1 Exercises

1.1 Projection onto a Convex Set

Let V be a Hilbert space and $M \subset V$ (a subspace). For $u \in V$, an element $\hat{u} \in M$ is called the best approximation of u in M if and only if

$$\|u - \hat{u}\| \leq \|u - v\|, \quad \forall v \in M$$

Now, we assume that $\mathcal{K}$ is a convex set:

1. Show that $\hat{u} \in \mathcal{K}$ is the best approximation of $u \in V$ if and only if:

$$(u - \hat{u}, v - \hat{u}) \leq 0, \quad \forall v \in \mathcal{K}$$

2. Show that $\hat{u}$ is unique.

3. Show that the operator $P : V \to \mathcal{K}$, $u \mapsto \hat{u}$, has the following properties:

a. P is monotone, i.e., $\langle Pu - Pv, u - v\rangle \geq 0, \quad \forall u, v \in V$.

b. P is contractive, i.e., $\|Pu - Pv\| \leq \|u - v\|, \quad \forall u, v \in V$.

1.2 Analytical Solutions to Variational Inequalities

In this exercise, we consider the obstacle problem:

$$\begin{cases} \text{Find } u \in \mathcal{K} = \{v \in H_0^1(\Omega) \mid v \geq \psi \text{ in } \Omega\} \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla(v - u)dx \geq \int_\Omega f(v - u)dx, \quad \forall v \in \mathcal{K} \end{cases} \tag{1}$$

We introduce $\sigma(u) \in H^{-1}(\Omega)$ defined by:

$$\forall \varphi \in H_0^1(\Omega), \quad \langle \sigma(u), \varphi\rangle = \langle f, \varphi\rangle - \langle \nabla u, \nabla \varphi\rangle \tag{2}$$

1. Suppose that:

$$\Omega =]0, 2[, \quad f = -8 \quad \text{and} \quad \psi = -3$$

i) Compute $u(x)$.

ii) Compute $\sigma(u)$ and verify that:

$$\sigma(u) = f\mathbf{1}_{\mathcal{C}(u)} \quad \text{in } \Omega$$

where $\mathbf{1}_{\mathcal{C}(u)}$ denotes the characteristic function of the set $\mathcal{C}(u)$, i.e.,

$$\mathbf{1}_{C(u)} = \begin{cases} 1 & \text{if } x \in \mathcal{C}(u) \\ 0 & \text{if } x \notin \mathcal{C}(u) \end{cases} \tag{3}$$

2. For:

$$\Omega =]0, 2[, \quad f = -16 \quad \text{and} \quad \psi = -2$$

a. Compute u.

b. Consider a uniform discretization with step $h = 0.5$. Compute u_h with $V_h = V_h^1$ ($\mathbb{P}_1$-Lagrange).

c. Verify that if $x \notin C(u_h)$, then $(u_h - \psi_h, -\Delta u_h - f) \neq 0$.

3. For $\Omega = (0, 1)$, $f =$ constant ≤ -8, and $\psi = -1$:

a. Show that the contact zone is of the form $[\xi, 1 - \xi]$, with $\xi \in (0, 1)$.

4. Suppose that:

$$\Omega =]-1, 1[, \quad f = 0 \quad \text{and} \quad \psi = \max\{0, 1 - \alpha|x|\}$$

where $\alpha > 0$ is large (i.e., $\alpha >> 1$). Then, show that:

i) $u(x) = 1 - |x|$.

ii) $\mathcal{C}(u) := \{x \in \Omega; \quad u(x) = \psi(x)\} = \{0\}$.

5. Suppose that:

$$\Omega =]0, 2[, \quad \psi = 0 \text{ and } f_\varepsilon = \begin{cases} 1 + \varepsilon & \text{if } x \in [0, 1] \\ -1 + \varepsilon & \text{if } x \in [1, 2] \end{cases}$$

where $|\varepsilon| < 1$. Show that:

a. The exact solution of the problem is given by:

$$u^\varepsilon(x) = \begin{cases} \delta_\varepsilon x + \alpha_\varepsilon x(1 - x) & \text{if } x \in]0, 1[\\ \delta_\varepsilon x + \beta_\varepsilon (x - 1)(x - \beta_\varepsilon^{-1}) & \text{if } x \in]1, \xi_\epsilon[\\ 0 & \text{if } x \in]\xi_\epsilon, 2[\end{cases} \tag{4}$$

where $\alpha_\varepsilon, \beta_\varepsilon, \delta_\varepsilon$, and ξ_ε are constants to be determined.

b. Compute the limit $\lim_{\varepsilon \to 0} u^\varepsilon(x)$ and compare it with the solution for f_0.