Exercises

Ex.1. Provide a variational formulation for the following equation for u.

$$u'' - u = -f, \quad u(0) = 0, u'(1) = 1.$$

Solution: Multiply by test function v that satisfies $v(0) = 0$, integrate by parts to get

$$\int_0^1 u'v' + uv \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x + [u'v]_0^1$$

Then, applying the boundary condition we get

$$\int_0^1 u'v' + uv \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x + v(1)$$

Defining

$$a(u,v) = \int_0^1 u'v' + uv \, \mathrm{d}x, \quad F(v) = \int_0^1 fv \, \mathrm{d}x + v(1)$$

and

$$V = \{u : a(u,u) < \infty : u(0) = 0\}$$

the problem becomes to find $u \in V$ such that

$$a(u,v) = F(v), \quad \forall v \in V.$$

Ex.2. Consider the function:

$$u(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Why the function u is not in $W^{1,p}(-1,1)$ for any $1 \leqslant p \leqslant \infty$?
Solution: because $u'(x) = \delta(x) \notin L^p(-1,1)$

Ex.3. Consider the problem

$$\begin{cases} -u'' - k^2 u = f, & \text{in } (0,\pi) \\ u(0) = 0 = u(\pi) \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$ and $k^2 \in \mathbb{R}$.

a) Cast the problem in variational form, stating carefully the spaces employed.

b) For what values of k is the problem not well-posed? (Hint: take $f = 0$ and look for nonzero solutions.)

c) For small values of k, the problem is coercive. For large values of k, the problem is not coercive. For what value of k does the problem lose coercivity?

Solution: We consider

$$\begin{cases} -u'' - k^2 u = f, & \text{in } (0,\pi), \\ u(0) = 0, \quad u(\pi) = 0, \end{cases}$$

where $f \in L^2(0,\pi)$ and $k^2 \in \mathbb{R}$.

(a) **Variational formulation.**

Let

$$V := H_0^1(0,\pi) = \{v \in H^1(0,\pi) : v(0) = v(\pi) = 0\}.$$

Multiply the equation by a test function $v \in V$ and integrate over $(0,\pi)$:

$$\int_0^\pi (-u'' - k^2 u)\, v\, dx = \int_0^\pi f v\, dx.$$

Integrating the term with u'' by parts and using $u(0) = u(\pi) = v(0) = v(\pi) = 0$, we obtain

$$\int_0^\pi u'(x)v'(x)\, dx - k^2 \int_0^\pi u(x)v(x)\, dx = \int_0^\pi f(x)v(x)\, dx.$$

Thus, the variational formulation is:

Find $u \in V$ such that

$$a(u,v) = F(v) \quad \forall v \in V,$$

where

$$a(u,v) := \int_0^\pi u'(x)v'(x)\, dx - k^2 \int_0^\pi u(x)v(x)\, dx, \qquad F(v) := \int_0^\pi f(x)v(x)\, dx.$$

(b) **Values of k for which the problem is not well-posed.**

We look for nontrivial solutions of the homogeneous problem ($f = 0$):

$$\begin{cases} -u'' - k^2 u = 0, \\ u(0) = 0, \quad u(\pi) = 0. \end{cases}$$

This is equivalent to

$$u'' + k^2 u = 0.$$

The general solution is

$$u(x) = A\sin(kx) + B\cos(kx).$$

From $u(0) = 0$ we get $B = 0$, and from $u(\pi) = 0$ we obtain

$$A\sin(k\pi) = 0.$$

For a nontrivial solution ($A \neq 0$), we must have

$$\sin(k\pi) = 0 \iff k\pi = n\pi, \quad n \in \mathbb{Z},$$

thus

$$k = n, \qquad n \in \mathbb{Z}.$$

Since the equation depends on k^2, the problem is not well-posed for

$$k^2 = n^2, \qquad n \in \mathbb{N}.$$

(c) **Loss of coercivity.**
For $u \in V$,

$$a(u,u) = \int_0^\pi |u'(x)|^2 \, dx - k^2 \int_0^\pi |u(x)|^2 \, dx.$$

By the Poincaré inequality on $H_0^1(0,\pi)$,

$$\int_0^\pi |u(x)|^2 \, dx \le \frac{1}{\lambda_1} \int_0^\pi |u'(x)|^2 \, dx,$$

where $\lambda_1 = 1$ is the first eigenvalue of $-\frac{d^2}{dx^2}$ with Dirichlet conditions. Hence

$$a(u,u) \ge (1-k^2) \int_0^\pi |u'(x)|^2 \, dx.$$

Therefore, $a(\cdot,\cdot)$ is coercive if $1 - k^2 > 0$, i.e.

$$|k| < 1.$$

The coercivity constant vanishes when

$$k^2 = 1.$$

Thus, the problem loses coercivity at $k^2 = 1$.

Ex.4. a) Formulate the following differential equation :

$$-u'' + u' + u = f, \quad u(0) = 0 = u(1).$$

as a variational problem on $H_0^1(0,1)$

b) Show that the bilinear form from this variational problem is coercive and bounded.

c) State and prove Céa's Lemma.

d) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0,1]$ with maximum width h. Let u_h be the solution to the Galerkin approximation of the variational problem using V_h, and let I_h :

$H^2(0,1) \to V_h$ be the interpolation operator onto V_h. Assuming $u \in H^2(0,1)$, and the following result,

$$\|u - I_h u\|_{H^1(0,1)} \leqslant h \|u''\|_{L^2(0,1)},$$

show that

$$\|u - u_h\| \leqslant Dh \|u''\|_{L^2(0,1)},$$

and provide a numerical value for D.

Solution: Multiplying by $v \in V$, integrating by parts and dropping the boundary terms due to the boundary conditions, we get

$$\int_0^1 (u'v' + u'v + uv)\,\mathrm{d}x = \int_0^1 fv\ \mathrm{d}x$$

First check F is continuous:

$$\begin{aligned} [\text{ Cauchy-Schwarz}]\ |F(v)| &\leqslant \|f\|_{L^2}\|v\|_{L^2}, \\ [\text{definition of } H^1 \text{ norm}]\ &\leqslant \|f\|_{L^2}\|v\|_{H^1}. \end{aligned}$$

Now check continuity:

$$\begin{aligned} [\text{Triangle inequality}]\ |a(u,v)| &\leqslant |(u,v)_{H^1}| + \left|\int_0^1 u'v\ \mathrm{d}x\right| \\ [\text{Cauchy-Schwarz}]\ &\leqslant \|u\|_{H^1}\|v\|_{H^1} + \|u'\|_{L^2}\|v\|_{L^2} \\ [\text{definition of } H^1 \text{ norm}]\ &\leqslant 2\|u\|_{H^1}\|v\|_{H^1}, \end{aligned}$$

so $C = 2$. Now check coercivity:

$$\begin{aligned} a(v,v) &= \int_0^1 (v')^2 + v'v + v^2\ \mathrm{d}x \\ [\text{completing the square}]\ &= \int_0^1 \underbrace{(v'+v)^2}_{\geqslant 0}\ \mathrm{d}x + \frac{1}{2}\int_0^1 \left((v')^2 + v^2\right)\mathrm{d}x, \\ &\geqslant \frac{1}{2}\|v\|_{H^1}. \end{aligned}$$

(d) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0,1]$ into elements with maximum width h. Let u_h be the solution to the the Ritz-Galerkin approximation of Equation (2) using V_h. Assuming the following result,

$$\min_{v \in V_h} \|u - v\|_{H^1_{[0,1]}} \leqslant h|u|_{H^2_{[0,1]}}$$

for $\gamma > 0$, show that

$$\|u - u_h\|_{H^1_{[0,1]}} \leqslant Dh|u|_{H^2_{[0,1]}},$$

and provide a numerical value for D.
Using the result of Part (a), plus $C = 2$ and $\alpha = 1/2$ from Part (b), we have

$$\|u - u_h\|_V \leqslant \frac{2}{1/2} \min_{v \in V_h} \|u - v\|_V$$

$$\text{[Error estimate]} \leqslant 4h|u|_{H^1_{[0,1]}},$$

hence $D = 4$. After integration by parts we get

$$\int_0^1 (u'v' + u'v + uv)\, \mathrm{d}x = \int_0^1 fv\, \mathrm{d}x - \beta v(1) + \alpha v(0)$$

The only change to the variational problem is that now

$$F(v) = (f, v) - \beta v(1) + \alpha v(0)$$

We have the trace estimate

$$|v(0)| + |v(1)| \leqslant \delta \|v\|_{H^1}$$

hence

$$|F(v)| \leqslant \left(\|f\|_{L^2} + \max(\beta, \alpha)\right) \|v\|_{H^1}$$

and hence F is still continuous.

Ex.5. Let V be the function space defined on $[0, 1]$ by

$$V = \left\{ u \in L_2 : \int_0^1 u^2 + (u')^2 \ \mathrm{d}x < \infty \right\}.$$

Consider the variational problem,

$$\text{Find } u \in V \text{ such that } \int_0^1 uv + u'v' \mathrm{d}x = \int_0^1 fv\ \mathrm{d}x, \quad \forall v \in V. \tag{2}$$

Let $0 < x_1 < x_2 < \ldots < x_{n-1} < 1$ define a subdivision of the interval $[0, 1]$. Let V_h be a finite dimensional subspace of V, consisting of all functions that are linear in each subinterval, and continuous between subintervals.

a) Formulate the finite element approximation for Equation (**??**) using V_h, and show how it results in a matrix-vector system of the form

$$K\mathbf{u} = \mathbf{F}.$$

[You do not need to compute the entries of K and $\mathbf{F}$, just provide a general formula for how they are calculated]

b) For the finite element approximation to Equation (**??**) given above, show that

$$\sum_{ij} K_{ij} = 1.$$

Solution: Let $\{\phi_i\}_{i=1}^m$ be a basis for S. Expanding u and v in the basis with coefficients u_i, v_i respectively, Equation (**??**) becomes

$$\sum_i v_i \left(\sum_j \int_0^1 \phi_i \phi_j + \phi_i' \phi_j' \mathrm{d}x u_j - \int_0^1 \phi_i f \ \mathrm{d}x \right) = 0$$

Since this must be true for all basis coefficients v_i, we have

$$\sum_j \underbrace{\int_0^1 \phi_i \phi_j + \phi_i' \phi_j' \mathrm{d}x}_{=K_{ij}} u_j = \underbrace{\int_0^1 \phi_i f \ \mathrm{d}x}_{=F_i}$$

which takes the required form.
(b) For the finite element approximation to Equation (**??**) given above, show that

$$\sum_{ij} K_{ij} = 1$$

Since the global nodal basis satisfies $\sum_{i=1}^m \phi_i = 1$, we therefore have $\sum_{i=1}^m \phi_i' = 0$. Hence,

$$\begin{aligned} \sum_{ij} K_{ij} &= \sum_{ij} \int_0^1 \phi_i \phi_j + \phi_i' \phi_j' \mathrm{d}x \\ &= \int_0^1 \left(\sum_i \phi_i \right) \left(\sum_j \phi_j \right) + \left(\sum_i \phi_i' \right) \left(\sum_j \phi_j' \right) \mathrm{d}x = 1. \end{aligned}$$