Exercises

Ex.1. In lectures we proved that for a stable discretisation of a stable (noncoercive) problem of the form

$$a(u,v) = F(v) \text{ for all } v \in V \tag{T}$$

a Galerkin approximation satisfies the quasi-optimality result

$$\|u-u_h\|_V \leqslant (1+c) \inf_{v_h \in V_h} \|u-v_h\|_V$$

where u_h is the solution to the Galerkin approximation of (T) over a closed subspace $V_h \subsetneq V$. Here $c = C/\tilde{\gamma}$, where C is the continuity constant of a and $\tilde{\gamma}$ is the discrete inf-sup constant.

(i) Prove that (under the same conditions) the Galerkin approximation is stable, i.e. u_h satisfies

$$\|u_h\|_V \leqslant c\|u\|_V$$

for the same constant $c = C/\tilde{\gamma}$.

(ii) For fixed $V_h \subsetneq V$ and a, consider the operator $P : V \to V_h$ defined by

$$a\left(u_h, v_h\right) = a\left(u, v_h\right) \quad \text{ for all } v_h \in V_h$$

In this equation we think of u as an input and $u_h = Pu$ as an output. Prove that P is linear and is a projection, i.e. $P^2 = P$.

(iii) A result from functional analysis states that for a bounded linear projection

$$P : V \to V$$

satisfying $0 \neq P^2 = P \neq I (I$ the identity operator on $V)$,

$$\|P\|_{\mathcal{L}(V,V)} = \|I-P\|_{\mathcal{L}(V,V)}$$

where the $\|\cdot\|_{\mathcal{L}(V,V)}$ norm is the operator norm

$$\|Q\|_{\mathcal{L}(V,V)} = \sup_{\substack{u \in V \\ u \neq 0}} \frac{\|Qu\|_V}{\|u\|_V}$$

Using this result, derive the improved quasi-optimality estimate

$$\|u-u_h\|_V \leqslant c \inf_{v_h \in V_h} \|u-v_h\|_V$$

Ex.2. Let $V = H_0^1(\Omega;\mathbb{R}^n)$ and $Q = L_0^2(\Omega)$. Let

$$L(u,p) = \frac{1}{2}\int_\Omega \nabla u : \nabla u \ \mathrm{d}x - \int_\Omega f \cdot u \ \mathrm{d}x - \int_\Omega p \nabla \cdot u \ \mathrm{d}x$$

We say (u,p) is a saddle point of L iff

$$L(u,q) \leqslant L(u,p) \leqslant L(v,p), \quad \forall v \in V, q \in Q$$

Show that $(\ u,p \ )$ is a weak solution of the Stokes equations if and only if it is a saddle point of the Lagrangian. (This is why these problems are called saddle point problems!)

Ex.3. Consider the mixed Poisson equation: find $(\sigma, u) \in H(\mathrm{div},\Omega) \times L^2(\Omega)$ such that

$$\int_\Omega \sigma \cdot \tau \mathrm{d}x - \int_\Omega \nabla \cdot \tau u - \int_\Omega \nabla \cdot \sigma w \ \mathrm{d}x = -\int_\Omega f w \ \mathrm{d}x$$

for all $(\tau, w) \in H(\mathrm{div},\Omega) \times L^2(\Omega)$.
(i) Write the mixed Poisson equation as the derivative of a Lagrangian $L(\tau, w)$.
(ii) What constrained optimisation problem is encoded by this Lagrangian?